# NORMATIVE LOGIC AND ETHICS

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To KAETHE LORENZEN

# PREFACE

This book contains the John-Locke-lectures given during the academic year 67-68 in Oxford.

Chapter VI (Modal Logic) has been added and Chapter VII (Foundations of practical philosophy) has been considerably enlarged.

I am grateful to Prof. Marjorie GRENE for many improvements in the text.

Erlangen, September 1968 Paul Lorenzen

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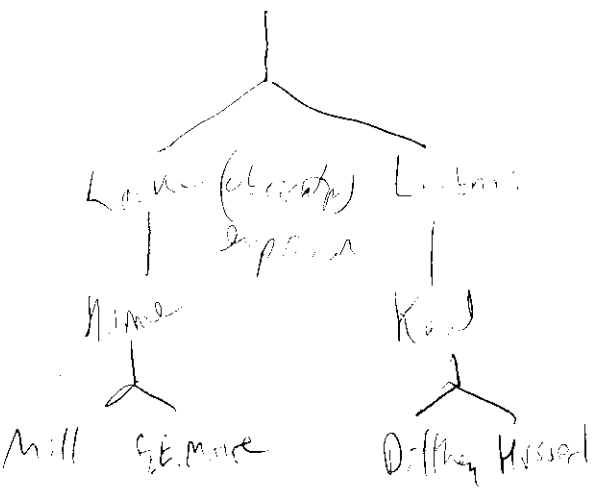
modern formal logic

Philosophy

Descartes – rationalist



# NORMATIVE LOGIC AND ETHICS

## *1. Elementary Sentences*

I am grateful to have the privilege of speaking to such a dinstinguished audience – the privilege of speaking as JOHN LOCKE lecturer in Oxford – the privilege of speaking about philosophy in this philosophical place κατ' ἐξοχήν.

Exactly 300 years ago philosophy was quite different at Oxford. JOHN LOCKE had just left Oxford – to be more precise, Christ Church College. In 1667 he was living in London, where he wrote his first essay on toleration and began preparing his famous essay concerning human understanding.

Philosophy was still taught in those days at Oxford primarily in the tradition of Occam, though there was a strong influence of Bacon and Hobbes. Bolder spirits, like LOCKE, even studied Descartes. This was the stream of thought which determined the course of history. LOCKE was the English response to the Cartesian challenge of making a philosophy out of the new exact science; LEIBNIZ was the German response. This dialogue between LOCKE and LEIBNIZ was continued by HUME and KANT – but as you know the dialogue deteriorated into monologues. In the 19th and 20th century we have, for example, MILL and G. E. MOORE on the English side, DILTHEY and HUSSERL on the German side. They seem to belong to different worlds. In the sciences the situation is different. There is no one who can compare to NEWTON – yet the English adopted the LEIBNIZian differentials, and the fundamental equations of electrodynamics are rightly called MAXWELL-HERTZ-equations.

Similarly, the rise of modern formal logic is due to the co-operation of FREGE and RUSSELL. The so-called elementary predicate calculus could rightly be called the FREGE-RUSSELL calculus.

In these lectures I am not going to talk about modern science. I shall try to philosophize instead. But what is that: "to philosophize"?

Let me begin with Locke's philosophy. He tried to understand modern science. As J. H. RANDALL formulates it: he "began with the rationalistic conception of Descartes and emerged with observationalism". Of course, this result of his philosophizing may be doubted but there is no historian of philosophy who seriously disagrees with the general appraisal of RANDALL: LOCKE was a good, honest, plain-spoken Englishman. The standard German history of philosophy, ÜBERWEG, says: "Er war ein wahrheitsliebender, vorsichtiger, nüchterner und frommer Charakter."

As far as I understand Oxford philosophy, these values are — fortunately — still in highest esteem, perhaps with one minor deviation. The highest praise today would be: "He was a good, honest, plain-meta-spoken Englishman."

That is the whole problem. Why can we no longer speak in a straightforward manner about, say, what we are doing, if we are doing science? This question can be answered, it seems to me, when we take into account that logic is an essential part of all sciences. The scholastics of the 15th and 16th century were in the same situation: they could not speak about anything without speaking about speaking. They knew, namely, that they should speak logically; and therefore, they tried first to fix logical rules. However this forced them into speaking about logic, that is, into metatalk.

Let us compare this with an argument of Mr. DAVID MITCHELL against the conventionalists (exemplified by Mr. STRAWSON). He attacks the claim that logical rules are based on linguistic rules. "Any attempt to *base* logical principles on something more ultimate, whether it be our system of contingent rules for the use of language or anything else, must be self-defeating. For the attempt consists of *deducing* conclusions from premisses and for deduction to be possible the prior validity of logical laws is a prerequisite."

There is an interesting note to this passage where MR. MITCHELL answers a "suggestion" to the effect that "to base" one thing on another does not mean "to deduce" the first from the second. He answers that he does not understand what can be meant, if not deduction.

Here now is the place where I should like to begin with apologizing for giving these lectures. How could I know if in this context "to base on" can mean something different from "to deduce from"? In questions about the English language I am — unfortunately — not at all competent.

Nevertheless I have the hope that we can come to some common understanding of words and things, even assuming my rather poor knowledge of English.

It seems to me that the difference of opinion between MR. MITCHELL and MR. STRAWSON may be described roughly – very roughly indeed – as the difference between the rationalist and the empiricist opinion. Locke's endeavour to overcome Descartes' rationalism was not a complete success. MR. STRAWSON asserts that logical rules can be *reduced* to linguistic rules – and these are conceived of as empirical facts. MR. MITCHELL wants to know what this "reduction" is, if not a logical deduction. As he does not get an answer, he asserts that at least some logical rules are "absolute" – a *pre*condition for the acquisition of any empirical fact. This I should like to call rationalistic.

Though this sounds rather hopeless, there remains the possibility that some linguistic rules are logical rules. Then the disagreement between rationalism and empiricism could be formulated in terms of the following questions: 1. What are these logical linguistic rules? 2. Are these rules to be accepted because they are factually accepted by the fluent speakers of some or all natural languages, or are they to be accepted because it is reasonable to speak according to these rules, independently of their factual usage? We shall still have the two opinions: of the empiricist, who states the acceptance of logical rules as a matter of fact, and of the rationalist, who accepts logical rules as a matter of reason.

As the title of my lectures is: "Normative Logic and Etbics", you will guess that I shall argue for the rationalist side. But you may be assured that this style of "arguing" will not consist in "deducing". To argue in the way I propose, we shall have to look first for a common basis, from which we can agree to begin. As such a basis I propose the use of elementary sentences. Of course, we could even begin one step deeper, namely, with one-word-phrases which are used as commands. Examples are: "Stop!", "Faster!", "Silent!". Counter-examples are: "Yes?", "John!", "Ouch!". Having given these examples and counterexamples, I ask you, of course, to imagine appropriate situations. And in saying this, I expect that you understand the phrase "to imagine appropriate situations". The phrases which are used as commands I propose to call "imperative-phrases". Now you will observe that in talking about these one-word-phrases I have already reached,

on the metalevel, the type of expression which I call 'elementary sentences', namely:

"Silent!" is an imperative-phrase
"John!" is not an imperative-phrase.

The last sentence spoken was, of course, even more complicated since it was on the metametalevel. This very word belongs to a metametametalevel. In this way we can easily obtain an infinite progression – but this is not an infinite *re*gression. If you prefer, I need not introduce the word "imperative-phrase" – nevertheless, we could come to a common understanding of imperative-phrases. So let us not bother about possible extensions on the metalevel: let us work our way upwards with phrases applied to things, not words: that is, with phrases on the so-called object-level.

Now suppose there is more than one person who could be silent. Then if you want John to be silent, you may use "John! Silent!" as an imperative phrase. Let us distinguish the first word as a *proper name* from the second word as a *predicator*. If he obeys, it may be *reported* to you: "John is silent". It is such reporting phrases, which I shall call "elementary sentences". It will be sufficient to indicate that I am going to use the linguistic expression "elementary sentences" for phrases of the form:

$$S_1, \ldots, S_n \; \varepsilon \; p$$
$$S_1, \ldots, S_n \; \varepsilon' \; p$$

with proper names $S_1, \ldots, S_n$ and predicators $p$. $\varepsilon$ is used as the English "is", $\varepsilon'$ as the English "is not".

Obviously all of you have decided to use such elementary sentences rather frequently. But what about a person who refuses to use them? If he refuses to use words at all, he will be brought to a school for mutes. If he has a kind family, they will try to help him – but, of course, they cannot *argue* with him. If he accepts the use of imperative phrases, but refuses the use of elementary sentences, he will be treated as mentally retarded.

Why am I telling you this? The point is that this decision to accept elementary ways of speaking is not a matter of argument. It does not make sense to ask for an "explanation", or to ask for a "reason". For to "ask" for such things demands a much more complicated use of language than the use of elementary sentences itself. If you ask such questions, in other words, you have already accepted the more elementary usage. So all of us do have a common

basis; we have all accepted at least the use of elementary sentences. Granted, it may be the case that some patterns of behaviour, which all people of a certain group have accepted, are rather stupid. For example, non-philosophers like to argue that philosophizing is a stupid activity. But no one can argue that the use of elementary sentences is stupid. Of course, it is easy to say: "Oh, how ridiculous that I am talking!", but then this should be one's last utterance. Since argument depends on the use of elementary sentences, one cannot *argue* that the use of elementary sentences is stupid.

So we who have accepted the use of elementary sentences may agree to call this usage "non-stupid" or "reasonable". This does not mean anything more than to confirm our decision, but we have the consolation that those who silently, without speaking, look at our behaviour cannot even think that we are stupid. For if we use the term "to think" in the sense of Plato, this would mean that they would speak to their own soul, using the elementary sentence "They are stupid".

What now about this playing with the phrases "to accept a usage" and "a usage is stupid"? Have we agreed to the rule that we will not accept stupid usages? No, I did not yet propose this rule to you. But I should now like to do so, or let me rather propose an affirmative rule: $U \varepsilon$ accepted $\Rightarrow U \varepsilon$ reasonable.

(In moral philosophy this "subjective" notion of reason will be modified.)

If we have accepted a usage, we call this usage "reasonable". The word "reasonable" does not make any difference. The decision always has to be made; namely, whether to accept or not to accept the usage, perhaps a new use.

In the case of elementary sentences nothing has to be said. We do not persuade our children to begin to speak with the assertion "To speak is reasonable", but simply by doing it, so that they may imitate us.

Only after having accepted a certain number of linguistic patterns, can we use these patterns "to argue". In the case of elementary sentences, we use proper names for naming objects and we use predicators to assert or deny properties of the named objects. But why do we do this? "Because there are objects and because there are properties which the objects have or do not have." This would be a fairly usual answer. But how do we know this? Well, it is easy to prove such existential sentences, begin-

ning with "there are objects", "there are properties" by giving examples. "SOCRATES is a philosopher" proves that there are objects: namely, SOCRATES, and that there are properties: namely, to be a philosopher.

It will be obvious by now that we prove the assertion by using an elementary sentence. Our first question is: "Why do we use elementary sentences?"

We get no answer – and we need no answer. We do not prescribe that there are objects – but we recommend to our children the use of proper names. We do not prescribe that there are properties – but we recommend to our children the use of predicators.

To this distinguished audience I surely will not have to recommend the use of elementary sentences. I can only remind you that we all accepted this usage long ago. Once more, provisionally, we may call this usage "reasonable".

With linguistic rules of a particular natural language the situation is different. Surely all those who have consented to speak English at all say, for example, "Socrates does not write poetry", if they want to deny the predicator "writing poetry" of SOCRATES. It is a convention to use "to do" in such negative sentences, however, it is not a convention to use negative sentences. Why not a convention? Instead of answering this question, let me answer the question: "Why don't I call it a convention?" Well, I call a usage a convention if I know of another usage which I could accept instead. Instead of using "to do" in negative sentences, there would be the possibility of affirming predicators in the form $S \varepsilon p$ and of denying them in the form $S \varepsilon' p$.

However I do not know of another behaviour which could replace the use of elementary sentences. If I did not accept proper names and predicators, I would not know how to speak at all. Of course, I could happily be silent – for a while. But I know this and only this way of beginning to speak – namely, beginning with elementary sentences in any of the conventional forms. Each proper name is a convention (because I know many sounds I could use instead), but to use a proper name at all is not a convention: it is a unique pattern of linguistic behaviour. Therefore, I am going to call it "logical". The same is the case with predicators. Each predicator is a convention. This is shown by the existence of more than one natural language. But all languages do use predicators. This is a logical feature of our linguistic behaviour.

I assume that you are in the same situation as I: namely, of not knowing any "Ersatz" or different behaviour to replace the use of proper names and predicators. This means that we have the common basis of having accepted the use of elementary sentences as a reasonable usage. It does not mean that you have agreed to the assertion that there are objects which have properties, but it is now easy to get to this assertion. As we are already using proper names, we now introduce the term "object". We may then say that our proper names name objects. Of course, we cannot introduce the term "property" in this way because, if we are using a predicator, we just have the objects of which we are affirming or denying the predicator – but we do not have anything other than the objects and the predicator. Where is a property "represented" by the predicator? To the best of my knowledge: Nowhere.

So, why is it reasonable to say "$S$ has the property $p$" instead of saying "$S$ is $p$"? It does not look reasonable, because it seems to be merely longer. However there are intermediate forms of sentences which are all true if and only if "$S$ is $p$" is true and it is reasonable to use these intermediate forms. Let us begin with "It is true, that $S$ is $p$" – I believe I need not repeat the situations in which this latter form makes sense. Then instead of "It is true that $S$ is $p$", we may say "$p$ applies to $S$" – this is only a shift of emphasis to the predicator $p$. We may then say, a bit more explicitly, "the predicator $p$ applies to $S$".

Now comes the decisive step to "$S$ has the property $p$". This step is justified if we are aware of the conventional choice of the predicator. We could have chosen any other sound for the same purpose, namely, for the purpose of distinguishing objects by affirming or denying the sound of them. If I am saying "$S$ has the property $p$", I am not only just asserting "$S$ is $p$" or: "It is true that $S$ is $p$", but I am in the same moment reflecting on my predicating, confirming that $p$ applies to $S$, and, in addition, expressing that I am not interested in the particular predicator $p$ of the English language, but that I am interested in $p$ only insofar as I may substitute for $p$ any other sound with the same use.

I am sorry that it is such a long story to explain what I propose to be doing by saying "$S$ has the property $p$" – but this is quite frankly my proposal. In more usual terms, the step from "the predicator $p$ applies to $S$" to "$S$ has the property $p$" is an *abstraction*. In order to express the abstraction from the particularities of the

chosen predicator, the interest shifts to the use which might be performed with any other sound. In order to express this, I propose to say "$S$ has the property $p$" instead of speaking about the predicator $p$.

In general I propose to use a sentence of the form $A$ (property $p$) if I want to assert $A$ (predicator $p$) in abstraction from the particularities of the predicator $p$.

Let $q$ be any predicator which I am using synonymously with $p$, e. g. "rot" instead of "red", and let $A$ (predicator $p$) be a sentence such that:

$$A \text{ (predicator } p) \text{ iff } A \text{ (predicator } q)$$

e. g. "red applies to $S$" iff "rot applies to $S$" then I shall call, the sentence $A$ (predicator $p$) *invariant*, more precisely: invariant with respect to synonymity.

In the case of invariant sentences only, I propose to say "$A$ (property $p$)" instead of "$A$ (predicator $p$)". My claim is that this proposed way of speaking is fairly reasonable. I do not say that it is necessary. You may not be interested in the invariance with respect to synonymity at all; but if you are, it is convenient to use this proposed technique of abstraction. This technique is applicable whenever there is an equivalence relation "$\sim$", such as synonymity between parts of our sentences. If $x \sim y$ implies $A(x)$ iff $A(y)$, $A(x)$ will be called invariant with respect to $\sim$. I invent an *abstractor* $\lambda$ and write in the case of invariant sentences $A(x)$ only:

$$A(\lambda\, x) \text{ instead of } A(x)\,.$$

I shall say that the object $x$ *represents* the abstract object $\lambda\, x$. In the case of synonymity I am using the word "property" as such an *abstractor*. Each property is, thereby, an abstract object, represented by synonymous predicators. Using the word "property" in this way, you will admit that I have proven my case: namely, that there are objects which have properties.

The term "proposition" may also be introduced as an abstractor. However this presupposes that we extend the equivalence relation of synonymity from predicators to elementary sentences. This is no difficulty; different proper names are called synonyms if they are used to name the same object.

$$S\, \varepsilon\, p \quad \text{and} \quad T\, \varepsilon\, q$$

are synonymous if and only if $S$ and $T$ are synonyms and if $p$ and $q$ are synonymous.

The proposition that $S \varepsilon p$ is then represented, e. g. by the sentence "$T \varepsilon q$".

This procedure can obviously be extended to two-place sentences and then obviously to any number of places. For example if we come to two-place sentences, $S, T \varepsilon p$ and $T, S \varepsilon q$, they may be called synonymous if $p$ and $q$ are explicitly introduced as converse 2-place predicators.

Still more complications arise if we extend our elementary sentences in such a way that *descriptions* are allowed instead of proper names. Let me call the new elementary sentences "*semi-elementary*". Elementary sentences have the form:

$$S_1, \ldots, S_n \; c \; p$$

with $c$ for $\varepsilon$, $\varepsilon'$ and with proper names $S_1, \ldots, S_n$ and the predicator $p$.

By replacing any one of the "subjects" $S_1, \ldots, S_n$ by a variable $x$ we form the phrase

$$\iota_x S_1, \ldots, x, \ldots, S_n \; c \; p$$

(which we shall read *the* $x$ with $S_1, \ldots, x, \ldots, S_n \; c \; p$).

Such phrases may be called potential descriptions. They have the form $\iota_x A(x)$, where $A(x)$ is a sentence-form. These potential descriptions are called *proper* descriptions if and only if there exists exactly one object $S$ with $A(S)$.

I would like to call the other *potential* descriptions *pseudo-descriptions*. For example: from "$x$, England $\varepsilon$ king" we get

$$\iota_x \, x, \text{ England } \varepsilon \text{ king}$$

(read as: the king of England).

The problem is whether or not to introduce propositions represented by such sentences as e. g. "the king of England is old".

If any one really desires such propositions, I would recommend that we allow him to assert them. On the other hand we can obviously get along without them. Therefore, my proposal would be to use only proper descriptions in "meaningful" sentences, i. e. in sentences for which we are going to introduce propositions represented by them.

If $\iota_x A(x)$ is a proper description — and if we let $A(S)$ be true — it is reasonable to use this description as subject in another elementary sentence $B(\iota_x A(x))$ by using this sentence as synonymous with $B(S)$.

This usage for proper descriptions is once more a linguistic technique, which could be avoided, but which is convenient. In contrast to the technique of abstractors, the technique of description is rather unavoidable in practice. In practical life we cannot always introduce proper names. We use indicators such as "this", "you" and so on, and we use indicator-descriptions such as "my dog". Also, the simple sentence "the dog is barking" in its ordinary usage does not presuppose that there exists exactly one dog, only that there exists exactly one dog here and now in the situation in which the sentence is used. The definite article "the" in "the dog is barking" is not the logical operator $\iota$. It is rather an abbreviated indicator "this".

But let me repeat that I am not competent at all to assert anything about the English language. The purpose of talking about elementary sentences is only to perform a logical construction of these sentences. Obviously it is a logical reconstruction of what we have always done in natural languages, but this is merely a contingent fact. Only in the case of elementary sentences in the strict sence we have no choice; either we accept the use of proper names and predicators or we do not speak at all. In introducing abstractors and descriptions we can more or less deviate from traditional linguistic techniques. We will have to use our reason to decide which ways of speaking we will accept as *logical* ways of speaking. We will have to establish norms for our speaking. We may compare these with the factual usage in our natural language, but the factual will never give a justification. In the beginning we can have no theoretical justification, i. e. no justification by arguments — we have to take the risk of actually beginning. Then immediately after accepting elementary sentences, we have at once the choice of extending our linguistic means, e. g. by abstractors or descriptions, but this extension can be accomplished by speaking about our use of elementary sentences. We need not use abstractors or descriptions on the meta-level in order to introduce them on the object-level; however, if we reflect on what we have done, we can see that we have used some linguistic means in addition to elementary sentences in order to be able to *say* what the new techniques are. This irreducible factor, which has to be supplied in addition to elementary sentences, if we want to be able to reduce abstractors and descriptions, consists — as you will not be surprised to hear — of the logical particles, junctors and quantifiers. In English I shall use the words "and", "or", "if — then",

"not" as junctors; "for all", "for some" as quantifiers; but again we will have to establish, to justify, norms for their use.

The instruction for the use of abstractors needed the statement of the condition of invariance

$$\bigwedge_{x,y} \cdot x \sim y \wedge A(x) \rightarrow A(y) .$$

The instruction for the use of the description-operator needed the statement of the conditions of unique existence:

$$\bigvee_x A(x)$$
$$\bigwedge_{x,y} \cdot A(x) \wedge A(y) \rightarrow x = y .$$

As we see here, at least the quantifiers $\bigwedge$, $\bigvee$ and the junctors $\wedge$, $\rightarrow$ are needed.

But why should we accept the techniques of composing sentences with such particles? What rules should we accept and why should we accept just these rules? If in the English language certain rules, for example, about substituting the simple $A$ for not not $A$, were generally accepted, — I do not know whether this is the case — but in any case, this would only be a contingent fact. This alone would be no reason for me to accept the usage. The problem which we encounter here is the problem of a "foundation" for formal logic.

## 2. *Logical Particles*

In these lectures I am trying to come with you to a common understanding of certain sentences. The difficulty seems not only to lie in my difficulties to understand English — it seems to lie more in my use of the term "understand". I am using it as synonymous with the German "Verstehen". "Verstehen" is very fashionable in German philosophy nowadays — though mostly the Greek term "hermeneutics" is preferred. Hermeneutical philosophy does not look for theories, systems of true assertions, but tries to understand — to understand man, to understand the world, to understand art, to understand language, i. e. to understand everything.

In the last lecture we have — I hope — agreed that we understand at least elementary sentences, e. g.,

'SOCRATES is wise',

or semi-elementary sentences such as:

'The teacher of PLATO is wise.'
'SOCRATES has the property wise.'

To understand a sentence very often means to know how to use it, that is, to have the "know-how" of its use. But this is not understanding as hermeneutical philosophy understands man and his world. To understand a sentence in this philosophical sense of understanding means not the know-*how* of its use, but the know-*why*. Why do we use proper names? Why do we use predicators? Only after having made up our minds on these questions, may we say that we understand philosophically what we are doing when we use elementary sentences.

In this lecture I should like to proceed to non-elementary sentences. I am going to extend the class of sentences which we understand.

Let us take an example: "Logically true sentences are true in all possible worlds." Do you understand this sentence? Perhaps you do not believe it – but you understand it, do you not? Perhaps some of you may even understand this sentence in the philosophical sense, but I have to admit that I cannot understand this sentence with the means which I have developed so far. Strictly speaking, therefore, I cannot even know whether this sequence of words is a sentence at all. Up to now, I understand only elementary sentences and semi-elementary sentences which contain abstractors or descriptions.

Let us reflect on what I have just said: "Semi-elementary sentences contain abstractors or descriptions." Do you understand this? I do hope, of course, that you understand – because I would not have said it otherwise. However, now we have the rather odd situation that I ought not to understand this, because it is not an elementary sentence.

If I had said about one of my earlier sentences, e. g.

"My sentence contained an abstractor."
or "My sentence contained a description."

then I would be entitled to understand, since both are semi-elementary sentences with a description "my sentence" – and I will take the rest, "had a so-and-so", as a predicator. True, both sentences are on the metalevel, but they are semi-elementary nevertheless. The difficulty lies in the "or". If I say, "My sentence

had an abstractor or description", how could I answer the question, "Why am I using "or" here?" without using "or" on the metalevel? The usual question is: What does "or" mean? Obviously it is easy to learn the know-*how* of using "or"; all native speakers have learned it. But what about understanding philosophically what we are doing when we use such logical particles as "or"?

GÖDEL proved (1931) that in order to prove the consistency of certain axiomatic theories you must have a meta-theory which is in certain respects richer than the object theory.

Since then it has become the accepted style not to be critical of the linguistic means on the meta-level. If one talks on the meta-level, one normally says: "I am talking in a meta-language" — though it is all more or less deteriorated English.

Now, to the best of my knowledge, Gödel's proof is correct, but I have written a textbook on Metamathematics with the hidden purpose of showing that the so-called philosophical consequences which are normally drawn from it have nothing to do with it.

If namely I begin — as I propose to do, in the tradition of Cartesian doubt — to doubt whether I understand "or", then it is of no help to teach me the use of "or" with the help of meta-talk which contains "or". Thus, in our case it is reasonable to insist on a method of introducing "or" into that part of language which we are going to use — without already using logical particles on a meta-level.

The usual truth-table method does not seem to satisfy this requirement. The truth-table method begins with the assertion that all sentences are true *or* false. We can avoid this by giving the truth-table the form of rules. For "or" we have the following rules:

$$A \varepsilon T \,,,\, B \varepsilon T \Rightarrow A \vee B \varepsilon T$$
$$A \varepsilon T \,,,\, B \varepsilon F \Rightarrow A \vee B \varepsilon T$$
$$A \varepsilon F \,,,\, B \varepsilon T \Rightarrow A \vee B \varepsilon T$$
$$A \varepsilon F \,,,\, B \varepsilon F \Rightarrow A \vee B \varepsilon F$$

(with $T$ = true, $F$ = false).

Of course, we have to understand the $\ldots ,,\ldots \Rightarrow \ldots$ notation for rules, however, it is easy to learn what we have to do if we are to "follow" such rules by sufficient practice.

In the case of elementary sentences I mean by $(S \varepsilon p) \varepsilon F$ nothing more than $(S \varepsilon' p) \varepsilon T$ and correspondingly $(S \varepsilon' p) \varepsilon F \leftrightharpoons (S \varepsilon p) \varepsilon T$.

Let us call a sentence (if you prefer: the proposition represented by the sentence) *definite* if it is true or false. We can formulate this proposal without using "or" by proposing the following rules:

$$A \varepsilon T \Rightarrow A \varepsilon \text{ definite}$$
$$A \varepsilon F \Rightarrow A \varepsilon \text{ definite}.$$

I am not going to bother you with the question whether all elementary sentences are definite. It would be a reasonable move to restrict our attention to definite sentences — and it would follow — that, if we start with definite elementary sentences, all composite sentences are definite. For the composition of the latter we can use the junctors: $\wedge$, $\vee$, $\neg$, according to rules which correspond to the truth-tables.

This two-valued-approach to logic is called classical (although it is very anti-Aristotelian). The difficulty with the classical approach lies – as BROUWER discovered 60 years ago – in the quantifiers. We may introduce the some-quantifier (the so-called "existential quantifier") by the rule:

$$A(S) \varepsilon T \Rightarrow \bigvee_x A(x) \varepsilon T.$$

From this rule no one up to now has been able to prove that e. g., the sentence "some odd numbers are perfect" is true. For what we have here is no longer an elementary sentence. The symbol $T$ is not a predicator; it only looks like one. We do not understand an elementary sentence $S \varepsilon p$, unless we understand $S \varepsilon' p$ also. Predication is the decision between affirming or denying the predicator $p$ of the object $S$. However, to assert a some-sentence is something different, because we have to defend such an assertion according to the given rule: we have to name an object $S$. Therefore, the sequence of symbols $\bigvee_x A(x) \varepsilon' T$ (or $\bigvee_x A(x) \varepsilon F$) has no meaning up to now; it has no use that has been explicitly agreed upon.

So, let us introduce the assertion of $\neg \bigvee_x A(x)$ by the following proposal: If I assert this, you may assert the affirmative part, i. e. $\bigvee_x A(x)$ and then I shall ask you to defend it. If you succeed, I will have lost my assertion; otherwise, I shall have won. In case you do not know how to defend $\bigvee_x A(x)$, it is advisable for you not to challenge the original thesis $\neg \bigvee_x A(x)$.

I do not propose to call $\neg \bigvee_x A(x)$ true if *you* cannot refute the assertion of it: I would call it "true" only if *no one* could refute

it. This meta-dialogical assertion is, of course, rather difficult to defend. Strictly speaking, it has up to now no agreed upon meaning.

This shows that in dealing with quantified sentences the classical disjunction "either true or false" no longer applies. It is, however, possible to introduce a negation $\neg$ by fixing its diagolical use:

| Assertion | Attack | Defense |
|---|---|---|
| $\neg A$ | $A$ ? | |

with no response other than to counterattack $A$.
For the some-quantifier we have already fixed the following attack-defense-rule:

| Assertion | Attack | Defense |
|---|---|---|
| $\bigvee_x A(x)$ | ? | $A(S)$ . |

For the all-quantifier we may now propose the following attack-defense-rule:

| | | |
|---|---|---|
| $\bigwedge_x A(x)$ | $S$? | $A(S)$ . |

Here the opponent chooses an object $S$. If an all-sentence has been defended against an attack, this does not justify calling the thesis "true". I propose to call a composite sentence true if and only if it can be defended as a thesis against all "possible" opposition; but this proposal does not make sense as long as the rules of the dialogical game, the game of attacking and defending sentences, has not been determined in full detail. In order to do this, I begin by introducing junctors into this game. But we cannot use the junctors as they are determined by the truth-tables, because the use of quantifiers may yield sentences which have no truth-value.

For the adjunction $\vee$ (I use this term for the non-exclusive disjunction) we have in analogy to the some-quantifier two rules:

| | | |
|---|---|---|
| $A \vee B$ | ? | $A$ |
| $A \vee B$ | ? | $B$ . |

The proponent of $A \vee B$ may chose which part he will defend after the opponent has attacked by "?". For the conjunction $\wedge$ the attacking opponent shall have the choice of the left or right part, which the proponent has to defend. We indicate this choice by $L$ ?, or $R$? respectively; so we get two rules in analogy to the all-quantifier:

| | | |
|---|---|---|
| $A \wedge B$ | $L$? | $A$ |
| $A \wedge B$ | $R$? | $B$ . |

Up to now I have said nothing about the junctor "if – then", and I shall not assert anything about the English usage of these words. On the other hand, I have already used these words rather frequently. In classical logic it is usual to define a *junctor* $\rightharpoonup$ by

$$A \rightharpoonup B \leftrightharpoons \neg A \vee B .$$

Let me call the sentences $A \rightharpoonup B$ "conditionals". The term "implication" I should like to reserve for a relation between sentences.

For the dialogical game I introduce a junctor $\rightarrow$ by the following attack-defense rule:

$$A \rightarrow B \qquad | \quad A \quad ? \quad | \quad B .$$

In words: If $A \rightarrow B$ has been asserted, it may be attacked by asserting $A$; as defense $B$ may be asserted.
The converse conditional $\leftarrow$ may be defined by:

$$A \leftarrow B \leftrightharpoons B \rightarrow A$$

It has the following attack-defense rule:

$$A \leftarrow B \qquad | \quad B \quad ? \quad | \quad A .$$

The introduction of these 4 junctors $\wedge$, $\vee$, $\rightarrow$, $\leftarrow$ will look rather arbitrary, but I must confess that I do not know of anything less arbitrary. Our 4 junctors exhaust all "simple" junctors. By a "simple" junctor I mean such a junctor that its attack-defense rules contain in the attacks, as well as in the defenses, each partial sentence exactly once. We require also that there is always a defense because otherwise we would come to dialogical negation.

It is just our 4 junctors which satisfy these requirements. All other junctors are definable in terms of the simple ones, perhaps including negation. E. g.

$$A \leftrightarrow B \leftrightharpoons A \rightarrow B \;\dot{\wedge}\; A \leftarrow B$$
$$A \sqcup B \leftrightharpoons A \wedge \neg B \;\dot{\vee}\; \neg A \wedge B .$$

I still have not shown how a dialogue runs. Let me first give an example. The thesis may be

'All atheists are stupid or wicked.'

in symbols: $\bigwedge_x \cdot x \,\varepsilon\, a \rightarrow x \,\varepsilon\, s \vee x \,\varepsilon\, w .$

The dialogue begins with the assertion of this thesis, let us say, written down by the proponent on the right side of the blackboard

(divided by a double line‖). Then the opponent attacks the thesis and the dialogue continues by means of alternating moves of the players.

| | | | |
|---|---|---|---|
| | | ‖ | $\bigwedge x \cdot x \,\varepsilon\, a \rightarrow x \,\varepsilon\, s \vee x \,\varepsilon\, w$ |
| | Russell ? | ‖ | $R \,\varepsilon\, a \rightarrow R \,\varepsilon\, s \vee R \,\varepsilon\, w$ |
| $R \,\varepsilon\, a$ | ? | ‖ | ? |
| $(R \,\varepsilon\, a)$ | | ‖ | $R \,\varepsilon\, s \vee R \,\varepsilon\, w$ |
| | ? | ‖ | $R \,\varepsilon\, w$ |
| | ? | ‖ | |

An elementary sentence in parentheses indicates a verification of the enclosed sentence. If the asserted elementary sentence $R \,\varepsilon\, a$ had been falsified, the opponent would have lost the game.

In the last position of the game the proponent has to show that Russell is wicked. But let us assume that he cannot do this: then the proponent has lost the game. His thesis is not true.

In order to fix the rules of this game, I propose in a first attempt the following:

*Starting Rule*: The proponent begins by asserting a thesis. The players make their moves alternatingly.

*General Rule*: Each player may either attack a sentence asserted by his partner or defend himself against an attack by his partner.

I hope that you agree that this is a reasonable convention to follow in order to find out whether or not a thesis is true. Actually I have to define "true" as the defensibility of a thesis in this game against any opponent. As long as the thesis is composed without quantifiers, there are always only finitely many strategies for the players. We can decide if there is a winning strategy for the proponent; only then the thesis shall be called "true". If the elementary sentences that occur are definite, this notion of truth coincides with the classical one. The defensibility (i. e. the existence of a winning strategy) may not be decidable if quantifiers occur — but "truth" is to mean defensibility in all cases.

Before I ask you to accept this "dialogical" notion of truth, I must unfortunately complicate the situation a bit more. The "general rule" has to be modified for cases which are more complicated than our first example.

Let us consider a conditional of the 3rd degree:

$$A \rightarrow B \dot{\rightarrow} C \ddot{\rightarrow} A \,.$$

The dialogue may run as follows:

| | | | |
|---|---|---|---|
| 1 | | $A \to B \dot{\to} C \ddot{\to} A$ | |
| 2 | $A \to B \dot{\to} C$ ? | $A \to B$ | ? 2 |
| 3 | $A$ | | ? 3 |
| 4 | $(A)$ | | |

In this situation the proponent may think of defending his thesis by asserting $A$, but the opponent may—rightly I think—point out that he asserted $A$ only in the attack against $A \to B$; therefore, he may insist that the proponent *first* defend himself against this attack, i. e. that he asserts $B$.

This example shows that the following restriction of the general rule is justified:

Each player may either attack a sentence asserted by his partner or he may defend himself against the last attack against which he has not already defended himself.

This formulation is nearly final. It still has the disadvantage of allowing the opponent indefinite repetition of attacks. Thus the proponent could not win even in the simplest cases. This requires us to impose a limit on repetitions by the opponent. Still we have to avoid arbitrariness. This leaves only the possibility of letting the opponent choose such a limit himself. So let the opponent choose a number $m$ from 0, 1, 2, . . . The general rule then yields the further restriction that the opponent may attack a sentence at most $m + 1$ times.

With this restriction of the opponent we may formulate a rule for the end of the game in a simple manner by the following

*Winning Rule*: If the opponent cannot make any further move, the proponent has won.

If the proponent has succeded in reaching such a final position, the opponent cannot defend himself against an attack and he cannot attack any assertion of the proponent (because the proponent has already defended himself successfully against all attacks).

If such a final position can be reached, against the ingenuity of any opponent, I propose to call the thesis "true". I have simply to ask you to accept this dialogical notion of truth. In the case of elementary sentences we start with a common understanding of these sentences, and therefore with their truth. But for non-elementary sentences, composed with logical particles, we have to

agree to a reasonable use. The traditional use in natural languages is a rather dubious authority. For example, the double use of a negation sign sometimes means a strong negation, sometimes an affirmation. We have to decide for ourselves how to take it.

Admittedly, my proposal, with its attack-defense-rules, general rule and winning rule, may look a bit too complicated to be reasonable. In particular, the choice of a number as a limit of repetitions seems too complicated for a fundamental rule of logic As a matter of fact the use of such a number can be avoided. This can be done by a modification of the general rule which simplifies the game without affecting the defensibility of any thesis. This will be the final statement of the *General Rule*. For the proponent it reads:

1. The proponent may either attack a sentence asserted by the opponent or he may defend himself against the last attack of the opponent.

But for the opponent it contains, instead of a limit of repetition, a much stronger restriction:

2. The opponent may either attack the sentence asserted by the proponent in the preceding move or he may defend himself against the attack of the proponent in the preceding move.

That this simplification of the general rule does not affect the defensibility of any thesis is a logically composite metadialogical assertion. Since the meta-dialogue may be played with the unmodified general rule, there is no circularity here; but this meta-dialogue is too complicated to be dealt with in these lectures. The modification of the general rule serves only to simplify the dialogue; therefore I will continue to use it, although it could be dispensed with.

With the rules of the dialogical game we now have sentences at our disposal which are compounded by means of logical particles, using elementary sentences as their parts. For these new sentences we have defined "truth" as defensibility against every opposition.

The dialogical game leads us in a natural way to generalize this notion of truth to "implication". In order to do this, consider the development of a dialogue. It starts with asserting a thesis $A_0$.

If $A_0$ is a conjunction or adjunction, an all- or some-sentence, the attacks will be made by writing question marks, perhaps with $L$, $R$ or with the choice of an instance for a variable. If a negation

or a conditional has to be attacked, the opponent has to assert a sentence himself. In the course of the dialogue the proponent will have to defend himself, and unless he defends an elementary sentence — and thereby wins the game — he will have to assert a new sentence $A$. Let us assume that for some reason the players want to have a break. After the break the opponent will have to attack $A$, and the proponent may attack all sentences asserted by the opponent before the break. We may say that after the break a new game begins. The starting position now is determined not merely by the thesis $A$, but by the opponent's assertions, too. We may use the following notation

$$\begin{array}{l|l} A_1 & \\ A_2 & \\ \vdots & \\ A_n & \quad A\,. \end{array}$$

This is a generalization of the game which starts with a thesis only. I would like to call the sentences given in advance by the opponent *hypotheses*. If the thesis $A$ can be defended with the hypotheses $A_1, \ldots, A_n$ given, I propose to say that the hypotheses *imply* the thesis.

This is obviously the case if and only if the thesis:

$$A_1 \wedge \ldots \wedge A_n \rightarrow A$$

is true.

In the Platonic Academy, during the time of pre-Aristotelian logic, this game was played in a rather tricky way. The proponent began by asking a question: $A$ or not $A$? He persuaded the opponent to make a choice. Let us say the opponent asserted $A$. Then the proponent took as his thesis $\neg A$, but he did not say so. Instead he began to ask the opponent seemingly harmless questions: What about $A_1$? What about $A_2$? ... He tried to get a sufficient system of hypotheses. If he attained such a system, he performed the final stroke by saying: Well, my dear friend, now you have admitted $A_1, \ldots, A_n$; *therefore*, you must admit $\neg A$, the contrary of your original assertion. The problem which Aristotle discovered here was the problem of justifying this *therefore*. This led him to the discovery of formal logic.

For our lectures this means that in the class of logically composite *sentences* we can now arrive at the distinction between empirically and non-empirically true sentences. The non-empirical truths

will be logical truths, but this does not mean that we may not introduce later on still different sentences, and thereby come to truths which are neither empirical nor logical.

## 3. Logical Truths

By means of the dialogical introduction of the logical particles we are now in a position to understand composite sentences. This does not mean that we already understand what logic – in the sense of "formal logic" – is. If we want to know the truth of a composite sentence, we have to look at the defensibility of that sentence as a thesis; but we do not need to know any so-called "logical rules" or "logical truths". Indeed, we have still to discover a distinction which allows us to interpret these terms.

Let us consider some examples which will exhibit such a distinction. As the first example I will take:

"In all Bavarian lakes there are fishes".

I take this as an ordinary version of the following sentence:

$$\underset{\text{Bav. lake}}{\bigwedge x}\ \underset{\text{fish}}{\bigvee y}\ y \text{ in } x$$

A dialogue may run as follows:

| | | |
|---|---|---|
| | $\underset{\text{Bav. lake}}{\bigwedge x}$ | $\underset{\text{fish}}{\bigvee y}\ y$ in $x$ |
| Tegernsee ? | | $\underset{\text{fish}}{\bigvee y}\ y$ in Tegernsee |
| ? | | $S$ in Tegernsee |
| ? | | |

This dialogue concludes with an elementary sentence (of the form $S, T \varepsilon p$, with $S$ as a proper name for a fish, $T$ for Tegernsee and the predicator $p$ for "to be in").

Depending on the truth of the elementary sentence, this dialogue will be won or lost. Of course, if this dialogue has been won, we still do not know whether or not the thesis is true. There are quite a few lakes in Bavaria, any of which could be chosen by the opponent in his first attack.

As the truth of the thesis depends on the truth of at least one of its elementary parts, I shall call the thesis "empirically" true. This term obviously is merely traditional. We say that the truth of elementary sentences is "empirically known", if it is known at all. We say that we come to know the truth of an elementary sentence by experience, if we come to know the truth at all.

The Greek word ἐμπειρία and the Latin word experientia have the same radical,* per, which is also in the German word "Erfahrung", but we need not speculate about the activities or passivities by which we come to the truth of elementary sentences. If I should propose to call the true elementary sentences "empirically true", you would rightly object that there is no need for such a term. Those elementary sentences which are true are true and that is all there is to it. In the case of composite sentences the situation is different: I propose to introduce a term for the truth of those composite sentences whose truth is independent of the truth of their elementary parts. Though it may appear at first that the truth of a composite sentence always depends on the truth of its elementary parts, this is not the case if we understand "to depend on" correctly.

In order to see this point, I shall first take the simple exemple:

"If Oxford is a river, then Oxford is a river".

The dialogue runs as follows:

| | |
|---|---|
| $O\,\varepsilon\,r$ ? | $O\,\varepsilon\,r \rightarrow O\,\varepsilon\,r$ |

and the opponent, who has carelessly challenged the thesis, will have lost the game because he cannot defend the elementary sentence. And even if he could defend the elementary sentence he would lose the game; since the proponent could then defend himself with just the same elementary sentence. Let "$a$" be any elementary sentence. The thesis $a \rightarrow a$ is defensible independently of the defensibility of $a$.

The same is the case with many other theses, e. g. $\neg . a \wedge \neg a$. Here the proponent can use the following strategy:

| | | |
|---|---|---|
| 1 | | $\neg . a \wedge \neg a$ . |
| 2 | $a \wedge \neg a$ ? | $L$ ? 2 |
| 3 | $a$ | $R$ ? 2 |
| 4 | $\neg a$ | $a$ ? 4 |

(The number following the question mark indicates the line which is attacked). In this situation the opponent has to attack the elementary sentence $a$, which he himself has asserted in line 3. It is only fair, therefore, to give up – instead of finding out whether $a$ is true or false (assuming that $a$ is definite at all). This leads us to separate the following composite sentences from the rest: namely, those sentences whose truth can be defended in such a way that the proponent finally has to defend merely an elementary sentence which has been asserted by the opponent.

All sentences of the forms $a \rightarrow a$ and $\neg . a \wedge \neg a .$ satisfy this condition. Aristotle discovered the existence of sentences which are true in this special way by such examples as the famous:

> "If all Greeks are men and if all men are mortal, then all Greeks are mortal".

We need not interpret this as a conditional. I think it is nearer to Aristotle to interpret his syllogisms as implications. Two hypotheses are given and the thesis has to be defended in such a way that the proponent finally has to defend merely an elementary sentence which has been asserted by the opponent.

Such a strategy exists if we interpret a sentence of the form "all $p$ are $q$" as $\bigwedge_x . x \varepsilon p \rightarrow x \varepsilon q . .$ I shall show that the following dialogue can be won against any opponent:

| | | | |
|---|---|---|---|
| 1 | $\bigwedge_x \cdot a(x) \rightarrow b(x) .$ | ‖ | |
| 2 | $\bigwedge_x \cdot b(x) \rightarrow c(x) .$ | ‖ | $\bigwedge_x \cdot a(x) \rightarrow c(x) .$ |

The dialogue may run as follows:

| | | | | | |
|---|---|---|---|---|---|
| 3 | | $y$ ? | ‖ | $a(y) \rightarrow c(y)$ | |
| 4 | $a(y)$ | ? | ‖ | | $y$ ? 1 |
| 5 | $a(y) \rightarrow b(y)$ | | ‖ | $a(y)$ | ? 5 |
| 6 | $b(y)$ | | ‖ | | $y$ ? 2 |
| 7 | $b(y) \rightarrow c(y)$ | | ‖ | $b(y)$ | ? 7 |
| 8 | $c(y)$ | | ‖ | $c(y)$ | |

and the opponent gives up.

It sounds rather implausible that the first "logical implications" should have been such complicated cases; nevertheless we need not assume that Aristotle had in mind exactly our dialogical game. He may have had in mind such rules as "$p \Rightarrow q$" (from $p$ go to $q$) when he said "$q$ belongs to all $p$". I shall deal with rules of this form in a later lecture. Meantime, let us introduce officially the

term "logically true" for those theses which can be defended in this special way, without bothering at the moment about Aristotle or about the question whether "logically true" should by synonymous with "non-empirically true". I would like to propose only the following rule:

$A$ ε logically true $\Rightarrow A$ ε′ empirically true

If a thesis can be defended in this special way with a system of hypotheses given, I shall say that the hypotheses *logically imply* the thesis.

In order to study logical truth and logical implications, it is convenient to introduce a *formal* variant of the dialogical game. Instead of beginning with elementary sentences, we now begin with prime-formulae:

$$a, b, \ldots$$
$$a(x), b(y), \ldots$$
$$a(x, y), \ldots$$
$$\ldots$$

We form compound formulae by using the symbols:

$$\wedge, \vee, \rightarrow, \neg, \bigwedge_x, \bigwedge_y, \ldots, \bigvee_x, \bigvee_y, \ldots$$

The attack-defense-rules are the same as before; they have merely to be completed by a rule for prime formulae $p$

$$p \mid ? \mid$$

with ? as attack and no defense; but in the formal game, however, the proponent is never to be allowed to attack a prime-formula.

The *General Rule*, therefore, reads:

1. The proponent may either attack a composite formula of the opponent or he may defend himself against the last attack of the opponent.
2. The opponent may either attack the formula put by the proponent in the preceding move or he may defend himself against the attack of the proponent in the preceding move.

The *Winning Rule* in the formal game is the following:

If the proponent has to defend a prime-formula which has been put by the opponent, the proponent has won.

This game with formulae I shall call the *formal game*; the game with sentences may be called the *material game*. In order to play

the formal game, we need not understand what sentences are, what it means to assert a sentence, or what logical particles are. We may forget about all this completely. However, in order to understand the formal game, that is, in order to answer the question, why it is reasonable to spend our time with this game, we will have to remember that the formal game is a formalization of the material game. The material game has to be understood first, then it has to be formalized. The result is the formal game. With the formal game we are simulating material dialogues.

Once we have established the formal game as a reasonable, though auxiliary instrument for investigating material dialogues, we may investigate the formal game *as if* it were an end in itself. We will call formulae which can be defended against all possible opposition *logically true formulae*. If a formula $B$ is defensible as a thesis when some formulae $A_1, \ldots A_n$ are given as hypotheses, we will say that $A_1, \ldots, A_n$ *logically imply* $B$. I shall write this in the following way:

$$A_1, \ldots, A_n \prec B .$$

In this notation $\prec$ represents a two-place relation between a system of formulae and a formula. Of course, the system could be replaced by the formula $A_1 \wedge \ldots \wedge A_n$.

Formal logic is a rather ancient discipline, since it began with Aristotle, flourished with the Stoics, flourished again with the later Scholastics, and has come to bear fruit in our century. Aristotle investigated material dialogues in the Topics, but since his Analytics we find the logician concerned mainly with logical implications, especially with rules which yield new logical implications from given ones. Aristotle uses, for example, the rule of contraposition:

$$A, B \prec C \Rightarrow A, \neg C \prec \neg B$$

in order to derive further syllogisms from those which he called "perfect". The Stoics (or perhaps earlier the Megareans) added rules of transitivity such as:

$$A, B \prec C \,,\, A, B \prec D \,,\, C, D \prec E \Rightarrow A, B \prec E .$$

They did not start with syllogisms, but with logical implications of the logic of junctors, e. g.

$$A \rightarrow B, A \prec B .$$

These are the beginnings of logical calculi. After the medieval period LEIBNIZ was the first logician to look for a *complete* lo-

gical calculus. Completeness means that *all* logical implications should be derivable from the initial implications of the calculus by the rules of the calculus. BOOLE was the first to give, though in an artificial algebraic form, a solution for the logic of junctors. FREGE gave the first calculus for the classical logic of junctors and quantifiers. A completeness proof is due to GÖDEL.

In our century, however, difficulties began with BROUWER's critique of the classical *tertium non datur* used for quantificational logic. HEYTING gave a logical calculus for nonclassical logic, normally called "intuitionistic". I would like to use the term "constructive logic" instead, since intuitionism is only one variant within the broader field of constructive logic and mathematics. The problem of the completeness of the HEYTING calculus has not been settled along the lines of the traditional approach. The difficulty is simply this: one has to establish a notion of constructive logical implication (or constructive logical truth) without using the derivability of a calculus.

The notion of dialogical defensibility yields exactly what is wanted.

Let us consider the problem of double negation; does $\neg\neg a$ imply $a$? The classicist says "yes", the constructivist "no". The dialogical game says "no", also.

| | | | | | |
|---|---|---|---|---|---|
| 1 | $\neg\neg a$ | | | $a$ | |
| 2 | | | ? | $\neg a$ | ? 1 |
| 3 | | $a$ | ? | | |

The proponent has no move besides repeating his attack of $\neg\neg a$; but as this does not change the situation, he cannot win. Nevertheless, we can understand the classical rule: *duplex negatio affirmat* if we add the hypothesis $a \vee \neg a$. This classical hypothesis is never mentioned by the classicists; that is the whole trick of "classical" logic.

| | | | | | |
|---|---|---|---|---|---|
| 1 | $a \vee \neg a$ | | | | |
| 2 | $\neg\neg a$ | | | $a$ | |
| 3 | | | ? | | ? 1 |
| 4 | | $\neg a$ | | $\neg a$ | ? 2 |
| 5 | | $a$ | ? | $a$ | ? 4 |

The opponent has to give up.

In classical quantificational logic the formula $\neg \bigwedge_x a(x)$ implies $\bigvee_x \neg a(x)$. In the formal game $\bigvee_x \neg a(x)$ is *not* defensible

with $\neg \bigwedge_x a(x)$ given alone, but only with the additional classical hypotheses:

$$\bigvee_x \neg a(x) \vee \neg \bigvee_x \neg a(x)$$

and

$$\bigwedge_x \cdot a(x) \vee \neg a(x) .$$

The dialogue runs as follows:

| | | | | | |
|---|---|---|---|---|---|
| 1 | $\bigvee_x \neg a(x) \vee \neg \bigvee_x \neg a(x)$ | | ‖ | | |
| 2 | $\bigwedge_x \cdot a(x) \vee \neg a(x)$ . | | ‖ | | |
| 3 | $\neg \bigwedge_x a(x)$ | | ‖ | $\bigvee_x \neg a(x)$ | |
| 4 | | ? | ‖ | | ? 1 |
| 5 | $\neg \bigvee_x \neg a(x)$ | | ‖ | $\bigwedge_x a(x)$ | ? 3 |
| 6 | | $y$ ? | ‖ | $y$ | ? 2 |
| 7 | $a(y) \vee \neg a(y)$ | | ‖ | | ? 7 |
| 8 | $\neg a(y)$ | | ‖ | $\bigvee_x \neg a(x)$ | ? 5 |
| 9 | | ? | ‖ | $\neg a(y)$ | |
| 10 | $a(y)$ | ? | ‖ | $a(y)$ | ? 8 |

It is easy to see that the thesis could not be defended without the first hypothesis.

We have now to show that the dialogically defined logical implications are identical with those of the constructive logical calculi as proposed by HEYTING, GENTZEN and others. This is rather tedious, but not difficult in principle. It is easy to arrive at a *complete* logical calculus, if we ask ourselves which positions

$$\begin{array}{c|c} A_1 & \\ \cdot & \\ \cdot & \\ \cdot & \\ A_n & B \end{array}$$

of the formal game are defensible. We now write these positions in one line as:

$$A_1, A_2, \ldots, A_n \;\|\; B .$$

We then ask for a calculus which yields all the winning-positions.

We denote a system of hypotheses by $\Sigma$, and a system of hypotheses in which a formula $A$ occurs by $\Sigma(A)$. Then we have at least for all prime-formulae $p$ the following winning positions:

$$\Sigma(p) \;\|\; p .$$

These will be the initial positions of our calculus. We now ask for rules which lead from winning-positions to further winning-positions. These rules may be called "admissible" rules. We get admissible rules if we ask e. g. how a position

$$\Sigma \parallel A \to B$$

with a conditional as thesis can be defended. The opponent has only one attack

$$\begin{array}{l|l} \Sigma & A \to B \\ A\ ? & \end{array}$$

and there is this defense.

$$\begin{array}{l|l} \Sigma & A \to B \\ A\ ? & B \end{array}$$

This yields the following: If the position $\Sigma, A \parallel B$ is a winning position, then $\Sigma \parallel A \to B$ is a winning position also.

We have, therefore, derived the following admissible rule:

$$\Sigma, A \parallel B \Rightarrow \Sigma \parallel A \to B\,.$$

The situation is more complicated, if we ask how to defend a position in which a conditional occurs as hypothesis

$$\Sigma\,(A \to B) \parallel C\,.$$

The opponent has to attack $C$. Let us consider the case where the proponent attacks $A \to B$ as response

$$\begin{array}{l|l} \Sigma\,(A \to B) & C \\ \ldots\quad ? & A\ ? \end{array}$$

The opponent may only continue with

$$\begin{array}{l|l} \Sigma\,(A \to B) & C \\ \ldots\quad ? & A\ ? \\ \ldots\quad ? & \end{array}$$

*or* with

$$\begin{array}{l|l} \Sigma\,(A \to B) & C \\ \ldots\quad ? & A\ ? \\ B & \end{array}$$

If both these positions are winning-positions, $\Sigma\,(A \to B) \parallel C$ will be a winning-position also. We have, therefore, another admissible rule:

$$\Sigma\,(A \to B) \parallel A\,,\ \Sigma\,(A \to B), B \parallel C \Rightarrow \Sigma\,(A \to B) \parallel C\,.$$

Each of the logical particles $\rightarrow$, $\neg$, $\bigwedge_x$, $\bigvee_x$ gives us in this way two "admissible" rules. The junctors $\wedge$, $\vee$ give us three rules each, because either the opponent or the proponent has a choice of two attack-defense-rules. This gives us $14 = 4 \cdot 2 + 2 \cdot 3$ admissible rules altogether. With the initial positions $\Sigma(p) \parallel p$ and these 14 admissible rules, we have a *complete* logical calculus: The system $A_1, \ldots, A_n$ logically implies $B$ if and only if the position $A_1, \ldots, A_n \parallel B$ is derivable in this calculus. We can formulate this result as a completeness theorem

$$A_1, \ldots, A_n \prec B \Leftrightarrow \vdash A_1, \ldots, A_n \parallel B$$

with $\vdash$ for derivability in the above calculus.

The logical calculus which can be justified in this way is known in the literature as a GENTZEN calculus: namely, as the "intuitionistic $G$ 3" in KLEENE's textbook on metamathematics. KLEENE proves that this calculus is equivalent to the HEYTING calculus.

Philosophically there is no reason to start with the historical fact that HEYTING published a certain calculus or to look for an interpretation of that calculus. It is, however, reasonable to start with material dialogues, to formalize this game, to look for admissible rules for winning-positions; this procedure leads us directly to an interpretation of the GENTZEN calculus and then indirectly to an interpretation of the HEYTING calculus. I would claim, therefore, that the dialogical approach justifies the logical intuitions of BROUWER and HEYTING. Independently of this relation to the recent history of formal logic, moreover, we have justified the GENTZEN calculus $G$ 3 as a convenient tool for investigating the formal dialogical game.

The justification of classical logical calculi is more difficult, because we have to justify the admission of tertium-non-datur-hypotheses. There are, of course, many cases in material dialogues in which these classical hypotheses are true, and then we need no justification. However, in arithmetic we do not know whether or not they are true; therefore, we need consistency proofs, if we are going to use classical logical implications in arithmetic. By a consistency proof I mean here a proof which tells us which sentences are constructively true if we have proven sentences to be "true" with the help of classical logic. The simplest result, first attained by GENTZEN, is that all implications between prime sentences which can be classically proven to hold, hold also constructively.

I shall not go into such proofs in any detail, but I would like to remark that such metamathematical results have to be proven without using the doubtful tertium-non-datur. The metamathematical results are, therefore, constructively true. But in order to clarify the notion of a *constructive metamathematics*, it is necessary to clarify first the notion of constructive arithmetic. Though I have occasionally already used arithmetical sentences. I shall have to introduce this new kind of sentence methodically without presupposing any prior knowledge of arithmetic. Since FREGE and PEANO it is usual to represent arithmetic as an axiomatic theory, but we will have to justify the axioms as true sentences. They will turn out to be our first examples of true sentences which are neither logically nor empirically true.

## 4. *Non-empirical truths in the formal sciences*

Thus far I have used two pairs of traditional terms: empirical-logical and material-formal. The first distinction is justified because in the dialogical game with sentences composed of elementary sentences some truths are defensible independently of the defensibility of their elementary parts. This special defensibility I designate as "logically true". The interest in logical truth then leads to a game with non-sentences; namely, with newly introduced symbols, called formulae, This name "formula" suggests the use of the Aristotelian terms material-formal to distinguish the two games.

However, this does not exclude the search for *sentences* which are neither elementary nor composed of elementary sentences. If we find such sentences, we may logically compose them; that is, we may compose them by using the logical particles. We will thus arrive at a new dialogical game which is not the formal game; so we will have to call it a material game — but its non-compounded sentences, its *prime-sentences*, will not be elementary sentences. If true, they will not be empirically true.

Arithmetic will be our first example of such a new dialogical game. Analysis, i. e. the theory of real and complex functions, and Metamathematics will be other examples. It is customary to call these sciences "formal" because their sentences are usually called "formulae". I should like to follow this terminological

tradition. Nevertheless it is necessary to keep in mind that, in terms of the dialogical approach, these formal sciences are material games. The formal game uses the symbolism of prime-formulae $a$, $b(x)$, $c(x, y)$, . . . and the logical particles only. If a terminological distinction is necessary, I shall call the formulae of the formal game "logical formulae."

I should not mind calling logic a formal science; but our interest is now in the extra-logical formal sciences. I avoid calling these extralogical formal sciences "mathematics" because, while traditionally "mathematics" always includes geometry, geometry will turn out not to be a formal science. Thus I must attempt to get along without using the term "mathematics". Nevertheless, our notion of the formal sciences will coincide roughly with the traditional notion of logic and mathematics, if geometry is excluded.

In order to introduce arithmetic, I begin — as mankind did at least 10.000 years ago — with counting. To count means to invent a sequence of symbols called numerals and to use them in the well-known way which we call "counting" objects. The simplest sequence of numerals is made up with one symbol only, say

|, ||, |||, . . . (one, one-one, one-one-one, . . .).

The introduction of these numerals is independent of the existence of numeral words in one's natural language. Nor does the use of the stroke-numerals need to be called an extension of one's language; but surely it is a new use of symbols.

I assume that you accept, as I do, the use of stroke-numerals for counting purposes as a reasonable activity. The sequence of stroke numerals is constructed by the following rules:

$$\Rightarrow \ |$$
$$n \Rightarrow n\,|$$

with $n$ as an "eigenvariable": that is, as a variable for strings of symbols constructed by means of these rules themselves.

This becomes rather inconvenient when we come to longer strings of symbols, but we need not consider the devices for abbreviations at this stage of our discussion. In particular, I need not recall our usual figures 1, 2, 3, . . . of Indian origin, i. e. ⊸, ⊸⊸, ⊸⊸⊸ . . . For our philosophical purpose of understanding how and why we use arithmetical sentences, it is sufficient to consider the simple stroke-notation; but as we know that there are different notations for numerals, we will use the term "number" as an abstractor.

The *number* | | | can be said to be represented, e. g., by the figure 3.

We do not count objects as a mere pastime but in order to know whether the objects counted will be sufficient, say for a fair distribution. This means that we want to *compare* different groups of objects by counting them. Instead of comparing the groups, we count them and compare the numbers. The *order* of numbers is not determined merely by examples. Of course, we could give examples of pairs of numbers $m$, $n$ such that we want to say $m < n$. E. g. |, | | | and | |, | | | | | would be such pairs. Instead however, we give rules for *all* pairs $m$, $n$ such that we wish to say $m < n$. The rules I propose are:

$$\Rightarrow \quad |, n \,|$$
$$m, n \Rightarrow m \,|, n \,|.$$

If a pair $m$, $n$ is constructible according to these — as I am going to say for short — $<$-rules, we put $m < n$:

$$m < n \leftrightarrows \vdash m, n\,.$$

If we reflect on what we have done and ask ourselves: "How do we know that | | | < | | | | | ?", the answer has to be "because | | |, | | | | | is a constructible pair according to the $<$-rules". The next question, retrospectively, is: "Why do we accept the $<$-rules?" This is no longer a question of truth, but a practical question. No one is forced to accept these rules, but they are to be recommended if we wish to open a new field of symbolic activity which is sometimes useful in dealing with groups of objects. Only practice can teach the value of such a technique. Terminologically, I would like to say that the $<$-rules have a *pragmatic* justification.

Let us compare this answer with the axiomatic approach. Instead of the $<$-rules one states $<$-axioms:

$$\text{(T 1)} \qquad | < n \,|$$
$$\text{(T 2)} \quad m < n \rightarrow m \,| < n \,|$$

or, if you prefer, with universal quantifiers in front:

$$\wedge_n \cdot | < n \,|\,.$$
$$\wedge_{m,n} \cdot m < n \rightarrow m \,| < n \,|\,.$$

All true prime-sentences $m < n$ are now logically implied by these axioms. The question: "How do we know the truth of the axioms?" is not permissible; one refers vaguely to empirical verification or confirmations. Starting with the pragmatically justified $<$-rules, on the other hand we see that the $<$-axioms are indeed true, in

the sense that they can be defended as theses in a dialogical game. We have now constructibility sentences $\vdash m, n$ as prime-sentences. If they are attacked, they have to be defended by performing the construction.

There are infinitely many sentences which are true in this sense, i. e. defensible. And it is on this foundation that the problem of axiomatization is possible; that is, of finding a convenient system of true sentences such that *all* true sentences are logically implied by them.

One simple answer to this question is the following. We consider the false prime-sentences, i. e. the true sentences $\neg\, m < n$. All these sentences are logically implied by:

$$\text{(T 3)} \quad \neg\, m < |$$

and

$$\neg\, m < n \rightarrow \neg\, m\, | < n\, | .$$

Of course, we have first to assure ourselves that these new axioms are true. Why will no one be able to construct $m, |$ according to the $<$-rules? Because only pairs $|, n\, |$ and $m\, |, n\, |$ are constructible. These pairs have $n\, |$ as their second member, but $m, |$ has $|$ as its second member and we have $| < n\, |$.

In order to "prove" the second assertion it is sufficient to defend

$$\text{(T 4)} \quad m\, | < n\, | \rightarrow m < n .$$

The strategy is simply this: If the opponent has constructed $m\, |, n\, |$, he will have used the second $<$-rule (the pair $m\, |, n\, |$ is not an initial pair $|, n\, |$ because for the first members we have $| < m\, |$).

Now we add to (T 1) – (T 4) the principle of induction, i. e. for each formula $A(n)$ we take the formula:

$$\text{(T 5)} \quad A(|) \wedge \wedge_m . A(m) \rightarrow A(m|). \rightarrow \wedge_n A(n)$$

as an axiom.

These formulae are indeed true, i. e. defensible. A dialogue may run as follows:

| | | | | |
|---|---|---|---|---|
| 1 | $A(\|)$ | | | |
| 2 | $\wedge_m \cdot A(m) \rightarrow A(m\|)$ . | | $\wedge_n A(n)$ | |
| 3 | $\|\|\|$ ? | | $A(\|\|\|)$ | |
| 4 | ? | | $\|$ | ? 2 |
| 5 | $A(\|) \rightarrow A(\|\|)$ | | $A(\|)$ | ? 5 |
| 6 | $A(\|\|)$ | | $\|\|$ | ? 2 |
| 7 | $A(\|\|) \rightarrow A(\|\|\|)$ | | $A(\|\|)$ | ? 7 |

The opponent gives up, because he has to assert $A(|\,|\,|)$ which the proponent has to defend. Against other choices than $|\,|\,|$ ? in line 3 the defense is similar.

(T 1) – (T 4) together with the principle of induction (T 5) are a complete axiomatization of the $<$-arithmetic.

As GÖDEL has proved, if we admit addition and multiplication in the prime-sentences, a complete axiomatization is no longer possible. There are always formulae $A(n)$ such that $A(m)$ is logically implied for each $m$, but $\wedge_n A(n)$ is not logically implied by the axiom system. Nevertheless, $\wedge_n (An)$ will be true: to defend it means nothing more than to defend $A(m)$ with $m$ chosen by the opponent.

To prove such a result is rather difficult, but in these lectures only the philosophical questions are relevant. How and why do we get true arithmetical sentences? We have only to ask for the prime-sentences. We have seen that the prime-sentences $m < n$ have their origin in rules for constructing pairs of numbers. We can define equality and inequality in the following way:

$$m \neq n \leftrightharpoons m < n \vee n < m$$
$$m = n \leftrightharpoons \neg\, m \neq n\,.$$

The usual axioms for equality:

$$m = m$$
$$m = n \dot{\rightarrow} A(m) \leftrightarrow A(n)$$

are logically implied by our axioms.

In order to introduce addition, I shall give rules for constructing triplets of numbers; namely:

$$\Rightarrow m, |, m|$$
$$m, n, p \quad \Rightarrow m, n|, p|\,.$$

The justification for proposing (and accepting) these rules is again pragmatic. As all of us have practiced addition since elementary school, I shall take it for granted.

Once the rules are accepted, everything has to be proven. For addition we first have to show (which I shall not do here) that the third member is uniquely determined:

$$\vdash m, n, p \;\wedge\; \vdash m, n, q \rightarrow p = q\,.$$

Then we may define:

$$m + n = p \leftrightharpoons \vdash m, n, p\,.$$

For multiplication the procedure is the same. We begin with pragmatically justified construction rules for triplets and define the prime-sentences $m \cdot n = p$ by means of constructibility.

To defend a prime-sentence means to perform a construction. Even the numbers (or numerals, if you prefer) have to be constructed before arithmetic can begin. So "construction" marks the difference between arithmetical prime-sentences and elementary sentences with proper names and predicators.

In order to come closer to the traditional KANTian terms, I shall use "synthetical" instead of "constructive". At least, according to the dictionaries, both the Greek word "synthesis" and the Latin word "constructio" can be translated as "Zusammensetzung", "putting together".

To defend an arithmetical truth, we need not use the "empirical" truth of elementary sentences; we need only use "synthesis" according to pragmatically justified rules. Therefore, I propose to call arithmetical truths *synthetic* truths. They are non-empirical truths. If we follow the KANTian usage and call all non-empirical truths "a priori", we have with KANT the arithmetical truths as *synthetic a priori truths.*

Let me add that there is also an important similarity between the arithmetical truths and logical truths. Logical truths are defensible in the formal game. Nothing has to be known about the world except a game with uninterpreted symbols; therefore, logical truths are called formal truths. In arithmetic we have the same situation. Once the pragmatic justification of the construction rules by which we introduce the numerals and the prime-sentences has been understood, we may forget it. We then have a game with symbols before us. Arithmetical truths are true sentences in this game, beginning with constructible sentences. Therefore, I should like to qualify the "synthetic a priori" as "*formally* synthetic a priori".

This terminological proposal may not be quite satisfactory, but it suggests the distinction between four kinds of a priori truths:

formally analytic — formally synthetic
materially analytic — materially synthetic

As I offer in these lectures just four kinds of a priori truths, the terminology serves my purposes satisfactorily.

The material a priori will be dealt with in the next lecture. The term "formally analytic" I propose to use for such truths as

need for their defense not merely the logical rules, but also *definitions*. Definitions occur, of course, also in connection with elementary sentences. If we start with the two-place predicator "married", such that "$x, y \varepsilon$ married" is the standard version of "$x$ is married to $y$"—and if we introduce for legal purposes the predicator "bachelor" by defining:

$$x \varepsilon \text{ bachelor} \leftrightharpoons x \varepsilon \text{ male} \wedge \neg \bigvee_y x, y \varepsilon \text{ married},$$

we get the true sentence:

$$x, y \varepsilon \text{ married} \rightarrow x \varepsilon' \text{ bachelor}$$

(i. e.: If $x$ is married to $y$, then $x$ is not a bachelor).

An example in arithmetic would be:

$$z \varepsilon \text{ prime} \leftrightharpoons z \neq 1 \wedge \bigwedge_{x,y} . x \cdot y = z \rightarrow x = 1 \vee y = 1 .$$

A logical implication of this definition would be:

$$z \varepsilon \text{ prime} \wedge x \neq 1 \wedge y \neq 1 \rightarrow x \cdot y \neq z .$$

In order to defend the truth of these sentences, one has to know merely some definitions. One does not have to know whether the prime-sentences are elementary or arithmetical prime-sentences, and one need know nothing about their defensibility. When we substitute the *definiens* for the *definiendum*, the sentences become logically true. Such sentences I propose to call "formally analytic truths". Logical truths are a proper part of them. Formally analytic truths which are not logically true I would like to call "formally analytic truths in the strict sense". "Formally analytically true s. s." therefore means that the sentence is defensible in virtue of its logical form *and* at least one definition. The term "definition" has a long complex history. I shall use it here only in the narrow sense of explicit definition:

$$x_1, \ldots, x_n \varepsilon p \leftrightharpoons A(x_1, \ldots, x_n) .$$

The predicator $p$ is introduced by the rule that the left side has to be replaced by the right side.

This narrow sense excludes so-called "implicit definitions" (not to mention "real definitions"). Very often, when it is said that an axiom system "implicitly defines" its objects, this is only a myth. But there are more serious cases, e. g. if it is said, that a formula

$A(x)$ defines implicitly an object $y$ that satisfies the condition. Here I would prefer to define explicitly:

$$y \leftrightharpoons \iota_x A(x) .$$

The so-called implicit definition of a function $F$ by a formula $A(x, y)$ is also reducible to an explicit definition with a description:

$$F x \leftrightharpoons \iota_y A(x, y) .$$

There remains only one interesting case, the case of so-called inductive definitions. In arithmetic sequences $p_1, p_2, p_3, \ldots$ of predicators are often introduced by conditions:

$$x \varepsilon p_1 \leftrightarrow A(x)$$
$$x \varepsilon p_{n+1} \leftrightarrow B(p_n, x) .$$

Since DEDEKIND it has been customary to "prove" that there exists for each $n$ exactly one *set* $S_n$ such that

$$x \in S_1 \leftrightarrow A(x)$$
$$x \in S_{n+1} \leftrightarrow B(S_n, x) .$$

By this "proof" we could define $S_n$ explicitly with a description, but this "proof" uses a naive or formalized version of Cantorian set theory. I propose instead to introduce sets as abstracts represented by sentence-forms; I shall give some details later.

For "inductive definitions" the abstraction-theory of sets has the consequence that we have to establish the sentence-forms $x \varepsilon p_n$ as meaningful, independently of set-theory. The simplest way I know is to introduce the predicators by means of construction rules for true sentences.

$$A(x) \varepsilon T \Rightarrow (x \varepsilon p_1) \varepsilon T$$
$$B(p_n, x) \varepsilon T \Rightarrow (x \varepsilon p_{n+1}) \varepsilon T .$$

With these rules, to defend $x \varepsilon p_1$ means to defend $A(x)$, and to defend $x \varepsilon p_{n+1}$ means to defend $B(p_n, x)$. So by an application of arithmetical induction we get the result: that for each $n$ the thesis $n \varepsilon p_n$ has a meaning; it is determined how to defend it in a dialogue. Thus it is obvious that these predicators satisfy the initial conditions of so-called inductive definition. Moreover according to the terminology proposed, we now have, in all cases where an "inductive definition" has to be used, not analytic truths, but formally synthetic truths.

I want to conclude my remarks about the formal sciences by indicating how arithmetic can be generalized and extended without leaving the limits of formal truth. The generalizations are easy. Instead of beginning with the simplest construction possible, namely, the construction of the stroke-numerals, we may investigate constructions of arbitrary strings of symbols. In this way we get the theory of calculi or metamathematics. This last name indicates the main application of the general theory, in which we take as strings of symbols the sentences of particular formal theories.

Methodologically, the theory of calculi does not raise problems which we do not face already in arithmetic. But something new seems to happen if we extend arithmetic, from its elementary level, the level we have been considering so far, to higher levels, i. e. to the theory of real numbers, customarily referred to as analysis. The new phenomenon here is the use of *sets* of numbers as objects of the theory.

Without going into the details of real numbers, we can formulate the essential problem in the following way. We start with arithmetical sentence-forms. We define an equivalence relation between such formulae, $A(n)$, $B(n)$, by $\bigwedge_n \cdot A(n) \leftrightarrow B(n)$. By an abstraction with respect to this equivalence we proceed from the formulae to the sets represented. This is the abstraction-theory of sets mentioned above. This is the way all mathematicians use sets in practice, although their philosophy normally forbids them to admit this. We denote the set represented by $A(n)$ as $\epsilon_n A(n)$.
The $\epsilon$-relation between elements and sets is defined by

$$m \in \epsilon_n A(n) \leftrightharpoons A(m) .$$

As all sets have to be represented by a formula, all questions about the existence of sets of numbers are questions about the existence of formulae in the arithmetical language. For the purposes of analysis this language need not be specified; on the contrary it may always be beld open for the introduction of new prime-sentences.

Now let us have a sentence-form referring to numbers and sets of numbers: $B(n, S)$. If we quantify the set-variable $S$, we get new formulae, for instance

$$A(n) \leftrightharpoons \bigvee_S B(n, S) .$$

Is this a definition? In order to answer this question we have to fix a dialogical use of the definiens. To defend $\bigvee_S B(n, S)$, we have to name a set. We have therefore to decide which formulae

may be chosen for representing a set. Whatever we decide, it should be clear that the formula $A(n)$ cannot be admitted as representing a set. $A(n)$ cannot be admitted, since we are attempting to fix the range of the variable $S$. Only *after* we have fixed the range, $A(n)$ has been defined. Of course, it may be that we can define a formula $A'(n)$ without using quantified set-variables—and that we can prove:

$$A'(n) \leftrightarrow \bigvee_S B(n, S) .$$

But to use set-variables without fixing their range and then to use formulae with quantified set-variables in order to define sets is a vicious circle. Although POINCARÉ had discovered this mistake of CANTORian set theory, RUSSELL comitted the same mistake by adding the axiom of reducibility to his ramified theory of types.

Since then axiomatic set theories have been en vogue. One simply postulates the existence of sets in such a way that the comprehension axiom:

$$\bigvee_S \bigwedge_x \cdot x \in S \leftrightarrow A(x) .$$

holds for all formulae $A(x)$, including those with quantified set-variables. This is called "impredicative" comprehension. Instead of postulating the existence of sets, HILBERT proposed that we consider axiomatic set theories as a formal game; but he added that this game makes sense only if its formal consistency can be proven as a metamathematical theorem. Such a proof would give a constructive interpretation of axiomatic theory.

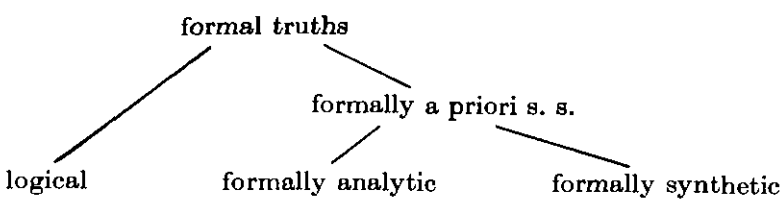
So there are two possibilities of giving a foundation to traditional analysis: either to restrict comprehension to predicative comprehension or to prove the consistency of impredicative axiomatic theories. I am in favour of a predicative analysis and have carried out this program in my book "Differential und Integral", but I would not mind if impredicative theories could constructively be proven to be consistent.

The so-called working mathematicians of our time prefer to work with impredicative theories without a consistency proof. To understand this unjustified preference is a matter of contemporary history. I shall not go into such an "explanation", because it is not relevant to the philosophical attempt to understand how formal sciences are "possible".

## *5. Non-empirical truths in the material sciences*

In the last lectures I indicated how formal sciences can be reconstructed. The aim was not to reconstruct uncritically the contemporary fashions of science, but to present a critical reconstruction. The result was a reconstruction of constructive logical calculi and of classical logical calculi with the help of a constructive consistency proof. I have indicated the beginnings of constructive arithmetic, of constructive metamathematics and of constructive analysis. Here the usual terminology is "predicative" analysis. The reconstruction did not allow the justification of impredicative analysis or, more generally, CANTORIAN set theory. These theories are to be considered as fashionable games only. Of course, a constructive interpretation may be found in the future, but this is at present merely wishful thinking.

In a refined version of KANTIAN terminology the true sentences of the formal sciences may be divided into three classes as shown by the following system:




The criterion for the distinction is the use of certain linguistic norms for verification: whether the rules of logic alone, logic and definitions alone, or logic, definitions and constructive rules are sufficient.

In all cases of formal truths we need not know anything about particular objects. Logic, definitions and symbolic constructions are to be justified pragmatically. This means that we have to understand our human situation in the world; we have to understand that the acceptance of linguistic norms is good for us; that we cannot become truly human without them. All this has to be understood even if we have no words to formulate it.

Yet, I repeat, we need know nothing about special objects. We use elementary sentences to speak about all objects whatever they may be. Then we form composite sentences out of elementary sentences by the use of logical particles. We may consider objects, i. e. things given by proper names, in whatever way we choose.

In what follows I shall deal with linguistic norms which are proposed for acceptance only because they deal with special objects. We are now looking more closely at the world, not merely realizing that there are objects which may be treated equally or differently by means of predicators. The result of speaking about what we see through this closer examination I shall call material sciences.

The first thing to do is to refine the method of speaking with elementary sentences only. If we have at our disposal nothing other than elementary sentences, we can determine the use of a predicator by examples and counterexamples only. I restrict myself to a one-place predicator $p$. The examples may be:

$$S_1 \varepsilon p, \ldots, S_m \varepsilon p$$

and

$$T_1 \varepsilon' p, \ldots, T_n \varepsilon' p$$

the counterexamples.

If we now come to a new object $R$, how are we going to decide whether $R \varepsilon p$ or $R \varepsilon' p$? It is an easy answer to say: "by immediate comparison", or to say: "in virtue of the similarity or dissimilarity with $S_1, \ldots, S_m$ or $T_1, \ldots, T_n$". Only if I am already able to use at least some predicators in new cases, am I able to understand such predicators as "to compare" or "similar".

Let me call the method of examples and counterexamples the exemplary determination of a predicator. This is approximately the same as what is usually called "ostensive definition", but it is not a definition in my strict sense and the examples may not be physical objects, as is normally assumed in so-called ostensive definitions. With examples and counterexamples we may also determine the use of predicators for our own activities, such as e. g. "naming" (sc. an object with a proper name) in contrast to "affirming or denying" (sc. a predicator of an object). To learn these predicators, it is useless merely to point to the physical events of uttering words.

Exemplary determination has an indirect variant: namely, telling stories. In St. Luke 10:29 Jesus is asked "Who is my neighbour?". St. Luke continues: "And Jesus answering said, a certain man went down from Jerusalem etc. etc." Jesus tells the story of the good Samaritan as an example of the phrase "to be neighbour unto someone". Though this is accomplished solely with words, no definition of the phrase is given. The words of the story are, of course, assumed to be already understood. I should like to call this method of exemplification by stories "indirect exem-

plary determination". If we avoid circularities it is a very powerful method of extending our common vocabulary.

Nevertheless, if we used only exemplary determinations, direct or indirect, for all our predicators, there would be too much disagreement, too much uncertainty. The use of some predicators may be made relatively stable by merely exemplary determination, but we have to look for another method of stabilizing the use of most of our predicators. I propose to use rules which prescribe a connection between different predicators. The simplest rules of this sort were dealt with by ARISTOTLE. His universal categorical sentences may be interpreted as the following rules for one-place predicators $p$ and $q$:

$$\begin{array}{ll} \text{affirmative:} & x \,\varepsilon\, p \Rightarrow x \,\varepsilon\, q \\ \text{negative:} & x \,\varepsilon\, p \Rightarrow x \,\varepsilon'\, q\,. \end{array}$$

These rules prescribe the transition from an elementary sentence of the form $x \,\varepsilon\, p$ to the sentence $x \,\varepsilon\, q$ or $x \,\varepsilon'\, q$, as the case may be. The question is: why accept such rules?

In order to answer this question, let me begin with an example of my own terminology. When introducing elementary sentences, I used two kinds of words, proper names and predicators. These terms are determined exemplarily. Thus we must imagine that up to now the use of these terms has been determined merely by giving some examples and counterexamples. This may lead to difficulties, e. g. someone may assert that the word "sun" is a proper name and also a predicator. A similar difficulty normally arises with the word "God". Now I hope that along with me you accept it as reasonable to fix the use of our terms in such a way that a proper name shall never be a predicator. Nothing other than this is proposed by the negative rule:

$$\text{(R)} \quad x \,\varepsilon\, \text{proper name} \Rightarrow x \,\varepsilon'\, \text{predicator.}$$

We do not have a vocabulary and a set of definitions such that the universal sentence: No proper name is a predicator (I take this as an ordinary version of

$$\bigwedge_x \cdot\; x \,\varepsilon\, \text{proper name} \rightarrow x \,\varepsilon'\, \text{predicator.})$$

is logically implied. This universal sentence is not a formally analytical truth. However, after having accepted the proposed rule, it is easy to defend this sentence:

| | | |
|---|---|---|
| | | $\bigwedge_x . x\ \varepsilon$ proper name $\rightarrow x\ \varepsilon'$ predicator. |
| Tommy | ? | Tommy $\varepsilon$ proper name $\rightarrow$ Tommy $\varepsilon'$ predicator |
| Tommy $\varepsilon$ proper name | ? | Tommy $\varepsilon'$ predicator |
| | ? | ( . . . ) |

The defense of the elementary sentence "Tommy $\varepsilon'$ predicator" can be performed by *deriving* it, beginning with sentences asserted by the opponent and using the accepted rule (R).

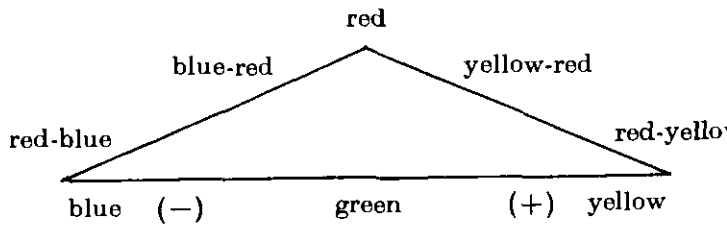
Let me call rules for predicators which are accepted for pragmatical reasons, namely, in order to improve the use of the predicators involved, *materially analytic* rules. If a sentence can be defended in virtue of its logical form, perhaps with the help of definitions and with the help of at least one materially analytic rule, this sentence will be called, a "materially analytic truth". Whether or not a sentence is materially-analytically true always depends on the rules accepted. The acceptance of the rules depends on our practice of using the predicators. We may say that the acceptance of the rules depends on "experience", but then "experience" is something different from our knowledge of empirically true sentences. We do not accept materially analytic rules because we experience the truth or falsity of elementary sentences.

I should like to illustrate this with the famous case of colour-words. Whereas the bull becomes nervous by the colour red, the philosopher since antiquity already becomes nervous by the word 'colour'. Let us assume that—independently of English usage—we have introduced "red" hy examples of fresh blood, and "greenish" by examples of different leaves. If we have a violet spot $S$ before us, why not call it "a greenish red", i. e. $S\ \varepsilon$ red $\wedge\ S\ \varepsilon$ greenish? This would be possible. However, if we have introduced a predicator "blueish" with suitable examples and counterexamples, it would be better to speak of "blueish red"—and to accept a rule which makes "red" and "greenish" contraries,

$$x\ \varepsilon \text{ red} \Rightarrow x\ \varepsilon' \text{ greenish} .$$

One could oppose accepting this rule by pointing to an orange spot, but, once more, it is possible to propose first the introduction of a new predicator "yellowish", and then to insist on the negative rule.

As an admirer of GOETHES *Farbenlehre* I should like to follow his proposals, i. e. to work with two basic colours and with red as the top-colour arranged in a triangle:

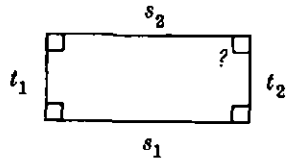



Within the pure colours blue is the darkest colour, yellow the lightest, and red is "the colour of the colours". Natural languages cannot decide anything about such questions. The physicist with his electrodynamic waves may be of some help, but he does not know anything about the "sinnlich-sittliche Wirkung", the sensuous-moral effects of the colours.

The different linguistic approaches to colours are of no philosophical interest in themselves. They are merely a convenient example of those linguistic conventions which I have called "materially analytic rules". For our philosophical purposes it is sufficient to know that sometimes it may be reasonable to propose and to accept such rules. For my own terminology I have already made use of this posibility more than once.

In our terminology we now have logical truths, analytic truths s. s. and formally synthetic truths. They all are non-empirical truths. In the following I wish to show that geometry provides us with an example of a priori truths of still another kind. I shall propose for them the term "materially synthetic", though the term does not matter at the moment.

"Geometry" is the traditional word for a theory which originated with THALES. EUCLID wrote its first classical text-book. However there is a long prehistory of geometry, because there is a prescientific language for our working with rigid bodies. This pregeometry is the basis of geometry for us. We begin with the exemplary determination of a pregeometrical vocabulary. Let us take body, side, edge, corner—and, e. g., with the help of a brick, the predicators: plane, straight, incident and vertical. Let us consider, in this pregeometrical way of speaking, a *plane* side with *straight* edges




Let $s_1$ be vertical to $t_1$, $t_2$ and let $s_2$ be vertical to $t_1$. What then about $s_2, t_2$. In Euclidean geometry we have, of course, the theorem:

$$s_1 \perp t_1 \wedge s_1 \perp t_2 \wedge s_2 \perp t_1 \rightarrow s_2 \perp t_2$$

But how do we justify this theorem—how do we justify teaching this to our children all over the world?

The empiricist says: "because in all instances of this general sentence, if the premisses were true, the conclusion has been found to be a true elementary sentence." Well, all red spots have been found to be not-green; all mammals have been found to have lungs. In the latter case, no one seriously proposes a *linguistic* rule

$$x \varepsilon \text{ mammals} \Rightarrow x \varepsilon \text{ having lungs}.$$

We have nothing more than an empirically true general sentence. In the case of the colours it is at least arguable to accept certain linguistic rules for colour words, so that, by this acceptance the general rules become materially analytic truths. But the case of geometry is different.

Nor is it of any help to say that our theorem is logically implied by axioms as stated by EUCLID or HILBERT. We should then have to ask about the truth of the axioms. It is only accidental that EUCLID did not choose our theorem as an axiom.

How, then, can our theorem be defended, if it is neither empirically true, nor formally true, nor materially-analytically true? Perhaps, and modern physicists seem to say so, it is not true at all.

In order to come to a reasonable opinion about this question, I should like to begin with PLATO. He made it clear that the points, lines and planes of geometry are something different from the corners, edges and sides of a body, of a *real* body, as we say, in order to distinguish it from *ideal* bodies. Nevertheless, the difficulty remains of justifying this Platonic talk of ideal bodies. In my answer I follow the line of KANTIAN philosophy, especially the "pragmatic idealism" of HUGO DINGLER, a German philosopher who lived from 1881—1954. "Idealism" in this sense starts with the

simple truth that a brick is not a natural object but an artefact. We have to *make* its side plane and we have to *make* it vertical. If we start inquiring how plane sides are made, we can easily go on from there to the art of telescope-making. Spherical lenses are produced by grinding two blocks of glass against one another. If you have sufficient patience, you can do it yourself without any machinery. Yet you will not get an *ideal* sphere, merely a *realization* of an *ideal* sphere. How can we understand these queer phrases of the lens grinders? Are they bad metaphysicians? Obviously they are not, though normally — with the exception of SPINOZA — lens grinders are not good metaphysicians either. They are not speaking merely about what they have made; they are also speaking about what they want to make. They have a norm which they try to fulfill. This norm is an *ideal* norm, which means that they will never fulfill the norm. So they seem to be very unreasonable.

This is just the question. Let me explain their ideal norm so that you may decide for yourself. The grinding-process *shall* make the sides of the lenses "homogeneous". This means that all corners — in geometrical language, we say "points" — shall be undistinguishable. The homogeneity, i. e. the undistinguishability of all points, means according to the usage of LEIBNIZ, that all sentence-forms which are true for one point of the sphere are also true for every other point.

Let us first agree on our geometrical language. We have — at least — two types of variables: $P_1$, $P_2 \ldots$ as point-variables and $S_1, S_2, \ldots$ as side-variables. As prime-sentence forms we have $P \mid S$ ($P$ incident with $S$) and $S_1 \perp S_2$ ($S_1$ vertical to $S_2$). All other sentences are logically compounded, with quantification allowed for both types of variables.

Now in this language the ideal norm for spheres can be formulated as the following principle of inner homogeneity:

$$P_1 \mid S \wedge P_2 \mid S \wedge A\,(S, P_1) \rightarrow A\,(S, P_2)$$

Here $A\,(S, P)$ is a sentence form which has no other free variables than $S$ and $P$, but it may have other quantified variables of both types.

I call this geometrical axiom-schema the principle of *inner homogeneity*, since for plane instead of spherical sides we have a second norm, the principle of *outer homogeneity:*

$$P_1 \dagger S \wedge P_2 \dagger S \wedge A\,(S, P_1) \rightarrow A\,(S, P_2)\,.$$

Here, we use the definition:

$$P \dagger S \leftrightharpoons \neg P \mid S$$

Spheres merely have to fulfill the first principle. Lens grinders try to fulfill the ideal norm of outer homogeneity by grinding three sides mutually against each other in order to realize planes.

What now about joining the lens grinders? No one has to join. But then he should not join the geometers either. This is at least my proposal for the term "geometry": it is the theory of ideal points and planes as they are determined by ideal norms.

Cutting two planes we get straight lines $k, l, \ldots$ .

Then we have to determine verticality by an ideal norm. We write $S \vdash l$ for $l$ being vertical to the plane $S$. EUCLID "defines" the right angle as being equal to its adjacent angle. We can generalize this idea and get the following principle of homogenetiy:

$$S \vdash l \wedge P_1 \mid S \wedge P_1 \dagger l \wedge P_2 \mid S \wedge P_2 \dagger l \wedge A(S, l, P_1) \rightarrow A(S, l, P_2)$$

I do not want to call these principles "definitions"; they are ideal norms which we have to accept in order to establish interpersonal cooperation, e. g. in architecture, fortification, and generally speaking in the *βίος ποιητικός*. Of course, these principles of homogeneity may at first appear arbitrary, but a closer examination will show you that each proposal of producing an *in*homogeneity would be arbitrary. The principles of homogeneity are simply the only possible way to avoid arbitrariness.

This last statement may be misleading. By the phrase "the only possible way" I do not wish to refer vaguely to a system of true sentences, such that all norms which are different from the principles of homogeneity are logically inconsistent with these sentences. The case is rather that the principles of homogeneity are one proposal which makes geometry possible; I simply do not know of any other proposal. And I am fairly sure that you do not know of any either. Thus I take the risk of beginning geometry with homogeneity because otherwise I should not know how to begin at all.

Taking the principles of homogeneity as axiom-schemata of a formal theory, we may now look for implied theorems. There is no reason to restrict "implications" to pure logic; we may also freely use arithmetic. However, we have first to show how numbers enter the scene. This is done by defining "measurement", beginning with the length of segments (of straight lines). Length is defined as a

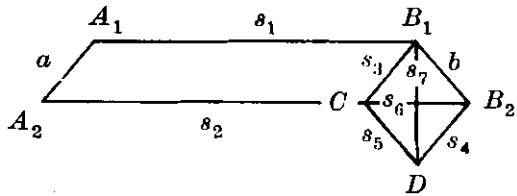
real number which represents a class of congruent segments. Therefore we have first to define congruence and to show that congruence is an equivalence relation, such that the abstract objects represented by congruent segments correspond uniquely to real numbers. That is the program of Descartes' analytical geometry. The problem is that for EUCLID (and for HILBERT) "congruence" is taken as a primitive term. In our geometrical language "congruence" has to be defined with the help of verticality. We may first of all define "parallel", either as EUCLID did,

$$S_1 \parallel S_2 \leftrightarrows S_1 = S_2 \vee \neg \vee_P P \mid S_1 \cup S_2$$

or more simply as

$$S_1 \parallel S_2 \leftrightarrows \vee_S \cdot S \perp S_1 \wedge S \perp S_2 .$$

Then we can "compare" two given segments $a$, $b$ by the following construction of auxiliary segments $s_1, \ldots, s_7$




These segments are in common geometrical notation determined as follows:

$$s_1 = A_1 B_1, A_2 \mid s_2 \parallel S_1, B_1 \mid s_3 \parallel a,$$
$$C = s_2 \cup s_3, B_2 \mid s_4 \parallel s_3, C \mid s_5 \parallel b,$$
$$D = s_4 \cap s_5, s_6 = B_2 C, s_7 = B_1 D$$

If and only if $s_6 \perp s_7$, the segments $a, b$ are defined as congruent.

The required properties of congruence so defined have to be proved from the principles of homogeneity. But obviously, this is impossible; we must at least add existence-axioms, e. g. that for each triplet of points there exists a plane incident with the point. In EUCLID such an axiom is called an *αἴτημα*, in contrast to a *κοινὴ ἔννοια*. The principles of homogeneity are hinted at in the *ὁρισμοί*.

Unfortunately the approach which I have indicated has not been published in detail. It is a program. But for our philosophical task of understanding how geometry is possible the principles of homogeneity already give, it seems to me, a sufficient answer.

All the discussion about "non-euclidean" geometries does not apply here, because they have "congruence" as a primitive.

After geometry I should like to sketch very briefly a program for *chronometry* as an extension of the program for geometry. Proceding from geometry to chronometry, we have to go from lens-grinders to the watch-makers. In order to get a criterion for an ideal clock, we have to establish the idea of *uniform* motion. A uniform motion has to be indistinguishable in all its points. The specific difficulty of chronometry is that we can compare at first only simultaneous motions. It is easy to define geometrically a ratio of velocities of two moving bodies for simultaneous positions. We call two motions "repeatable relative to each other", if they have for all repetitions the same ratio in every position. The ideal norm for uniform motion, then requires that the ratio of velocities of repeatable motions is constant and always the same independently of a temporal displacement (cf. P. Janich: Die Protophysik der Zeit, to be published by Bibl. Inst. Mannheim, 1969). In other words, the "similarity" of two motions, defined by a constant ratio of velocities, has to be preserved in every repetition and in every displacement of the beginning or ending of the motions. Each of two motions fulfilling these conditions is, therefore, indistinguishable in all parts, the criterion of distinction being the ratio of velocity at a point relative to a repeatable motion. Such a motion is uniform – and on the basis of this notion "time" becomes measurable.

The next step after chronometry is to make *materia* measurable. Following rather closely the historical origin of the notion of the inert mass, I would propose to begin with the collision of bodies. In the case of inelastic collisions we can measure the ratio of the velocities $v_1$, $v_2$ of the colliding bodies. (The velocities have to be measured relative to the bodies after the collision.)

We have to determine what an ideal (undisturbed) collision shall be, but, as we want to measure the ratio of the velocities and as we cannot measure the velocities themselves without introducing arbitrary units there is once more only one proposal which is reasonable, i. e. not arbitrary, that is, when the collision is repeated, the ratio shall always be the same.

The rest is mathematics. We may define the ratio of "masses" by

$$\frac{m_1}{m_2} \leftrightharpoons \frac{v_2}{v_1}.$$

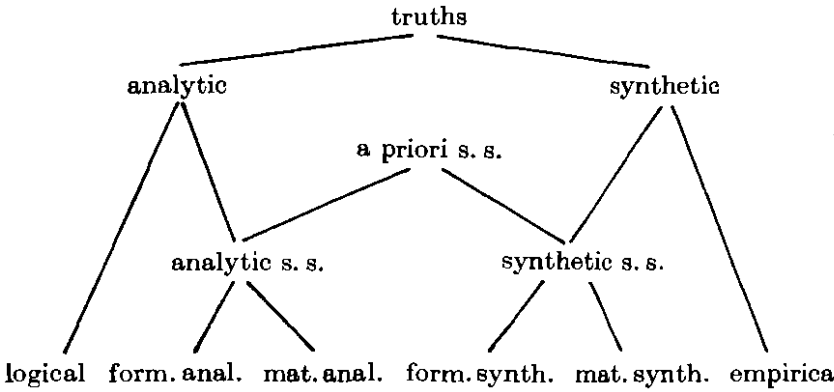
We may define "impulse" as $m \cdot v$, "force" as $\frac{d(m \cdot v)}{dt}$, etc.

By analogy to geometry and chronometry I should like to call this part of mechanics, which is concerned with the ideal norms for measuring materia, "*hylometry*" from the Greek word for materia. Hylometry states the ideal norms for collisions and thereby gives the first a priori causal relations ("causally necessary truths"). Hylometry coincides with classical mechanics without gravitation. This part is sometimes called "rational mechanics", but as it is usually presented it is difficult to understand the rationale of this discipline.

Geometry, chronometry and hylometry are a-priori theories which make empirical measurement of space, time and materia "possible". They have to be established before physics in the modern sense of an empirical science, with its hypothetical fields of forces, can begin. Therefore, I should like to call these three disciplines by the common name: *protophysics*. The true sentences of protophysics are those sentences which are defendable on the basis of logic, arithmetic and analysis, definitions *and* the ideal norms which make measurements possible.

These ideal norms are clearly different from the construction rules of arithmetic, nor are they definitions. They are not formal determinations, because we are no longer dealing merely with strings of symbols. We are dealing with materia, grinding its sides, regulating its movements and producing collisions. We prescribe by norms how the materia shall "behave", if I may use this biological metaphor. In contrast to materially analytic determinations, where we prescribe rules for our predicators in order to fit the world, we now force the materia to fit our ideal norms. In protophysics our relation to the world is no longer passive (with our activity merely on the linguistic side), we are now actively changing the world. In order to avoid a new term for expressing this difference, I wish to give the task to the traditional term "synthetic". By this terminological convention we get the result that these protophysical true sentences which are dependent on the ideal norms are to be called "materially synthetic" truths.

Assuming that at least some of you will look benevolently on my attempts to revise traditional terms, I should like to offer finally the following system of all the truths reconstructed in these lectures thus far:




## 6. Modal Logic

The final purpose of philosophy, as I understand it, is to make the art of practical reasoning teachable. But no one can know whether there is such a thing as "the art of practical reasoning" or, if there is, whether it can be made teachable, unless he tries both to practice it and to teach it. As a propaedeutical step toward finding practical uses of reason, however, it is useful to investigate theoretical reason first.

Since ARISTOTLE, the father of logic, there has been a tradition of using so-called "modalities" in the language of the sciences*. For example, the words "necessary" and "possible" are usually used in English as modalities. It is the task of modal logic to establish a reasonable use for these modalities. In the language of morals, law and politics we have the corresponding modalities "obligatory" and "permitted".

I am not going to investigate the actual usages of these English words. Instead, I will propose a use in such a way that it can be claimed that the proposed use reconstructs the intentions of the authors of texts on modal logic. I do not claim to reconstruct every such intention, since some traditions are inconsistent with one another; in such cases we have to come to a reasonable decision about which intention to reconstruct. Thus the teaching of THEOPHRASTUS, ARISTOTLE's successor, on modal logic already was incon-

* Modal logic was Aristotle's favorite child, as we can see from the proportion of his text devoted to modal logic in comparison with that dealing with the assertoric syllogistic.

sistent with that of ARISTOTLE. These inconsistent teachings have been carried through all our history, and even modern commentators (or historians of logic) such as BOCHEŃSKI and KNEALE fail to come to a decision on the matter.

However, in our century a new initiative has been taken by LEWIS with his invention of a system of calculi $S_1 - S_5$. These calculi were intended to be formalizations of an "intuitive" notion of necessity. Starting from the uninterpreted calculi, mathematical logicians such as BETH and KRIPKE have managed to "interpret" these calculi within a metalanguage, using only classical quantificational logic. But these "interpretations" are justified only insofar as they interpret the LEWIS-calculi. We see here the order in which formalistic logicians typically proceed: first invent any calculus (it is sufficient to refer vaguely to "reason" or "intuition" to make the invention acceptable), and then, using all one's ingenuity, look for an interpretation within modern mathematical theories. However, there is another way of doing modal logic, the way which is to be followed here.

Imagine a situation in which a system $\Sigma$ of sentences has been generally accepted as *true* by a certain group of people. In such a situation, all sentences logically implied by $\Sigma$ have a special importance, namely, they *should* all be generally accepted in the group. This importance justifies a special notation, "$\prec$", by which I will denote logical implication. (This is a two-place relation between sentences, not to be confused with the junctor "$\rightarrow$" in conditionals.) The symbol $\Delta_\Sigma$ (read as "necessary relative to $\Sigma$") is introduced by the following definition:

$$\Delta_\Sigma A \leftrightharpoons \Sigma \prec A$$

Here "$A$" is a variable for modal-free sentences.

By itself, this definition would not justify our setting up an extra theory, called "modal logic". But there are theorems about statements of the form $\Delta_\Sigma A$ which are true *independently* of the particular $\Sigma$. Because of this indepedence I will write $\Delta$ as an abbreviation for $\Delta_\Sigma$. Thus the following theorem is obviously true (as a theorem about logical implication):

$$\Delta A \wedge \Delta B \rightarrow \Delta (A \wedge B)$$

As this conditional is true for all $\Sigma$, I'll call it modal-logically true—and I'll say, that the premisses modal-logically imply the

conclusion. Using the same symbol $\prec$ as before (but now for modal-logical implication, too) we get

$$\Delta A \wedge \Delta B \prec \Delta (A \wedge B) .$$

The task of modal logic is to determine the class of modal-logical implications. Already in ARISTOTLE, Analytica priora, book 1, chapter 8, we find (if we are looking for it) the following rule:

$$A_1 \wedge A_2 \ldots \prec B \Rightarrow \Delta A_1 \wedge \Delta A_2 \wedge \ldots \prec \Delta B$$

The validity of this "Rule of ARISTOTLE" is obvious under the proposed interpretation. Moreover, the rule is convertible, because, if $\Delta_\Sigma A_1 \wedge \Delta_\Sigma A_2 \ldots \prec \Delta_\Sigma B$ holds for all sytems $\Sigma$ it holds especially for $\Sigma = A_1 \wedge A_2 \wedge \ldots$ and this gives immediately $\Delta_\Sigma B$, i. e. $A_1 \wedge A_2 \wedge \ldots \prec B$.

The theorem mentioned above: $\Delta A \wedge \Delta B \prec \Delta (A \wedge B)$ is, of course, a consequence of ARISTOTLE's Rule.

Besides this rule we find also in ARISTOTLE the trivial implication $\Delta A \prec A$ (necessity implies truth). This follows from the assumption that $\Sigma$ is a basis of *true* sentences. In modal logic we have to investigate, which implications hold between arbitrarily compound sentences. Besides the junctors and quantifiers the modality $\Delta$ may be used. Under the given interpretation $\Delta_\Sigma A \leftrightharpoons \Sigma \prec A$ the iteration of $\Delta$ does not make sense, because $\Delta_\Sigma \Delta_\Sigma A$ would be $\Sigma \prec (\Sigma \prec A)$.

Only as a formal game, the iteration of $\Delta$ may be admitted and then e. g. the arbitrary rule, to treat $\Delta\Delta A$ as equivalent with $\Delta A$ will easily occur to every mathematically trained mind. Also the false analogy with $\Sigma \rightarrow (\Sigma \rightarrow A)$ (which is equivalent to $\Sigma \rightarrow A$) may suggest to "postulate" the equivalence of $\Delta\Delta$ with $\Delta$.

Instead of playfully suggesting such postulates I'll restrict the use of $\Delta$ to its non-iterated use: only if $\Delta$ does not already occur in a formula $A$, $\Delta A$ shall be a formula.

But the logical particles may be applied without restriction. We have then to investigate how a compound sentence can be defended as a thesis, if some hypotheses are given. The admission of $\Delta$ leads to two new situations:

1. In the course of the dialogue the opponent may have put a formula $\Delta A$. If we would not have the restriction that $\Sigma$ has to be treated as unknown, the proponent would surely ask, relative to which $\Sigma$ the "necessity" is asserted. But in modal

logic, this is not allowed. So the proponent may only force the opponent to admit $A$, if he has admitted $\Delta A$.
This gives us the following attack-defense-rule for $\Delta$-formulae (*i. e.* formulae beginning with $\Delta$):

$$\Delta A \;\mid\; ? \;\mid\; A$$

2. If the proponent has put a $\Delta$-formula $\Delta B$ and if the opponent attacks this (by "?"), the proponent has to defend $\Delta B$ as logically implied by all formulae put by the opponent beforehand. Only the "Rule of ARISTOTLE" is available for defending $\Delta B$.

   This gives the following additional

   $\Delta$-defense-rule: If the proponent defends a $\Delta$-formula he may attack only the $\Delta$-formulae (the beginning $\Delta$ deleted) put by the opponent beforehand.

It is easy to extend the logical GENTZEN-calculus established in the 3. lecture to yield exactly all the winning-positions of this dialogical game with $\Delta$.

In order to defend a thesis $B$, if a system $S(\Delta A)$ of hypotheses is given which contains a $\Delta$-formula $\Delta A$, the proponent may attack $\Delta A$. The opponent has to defend himself by putting $A$. This gives the new position $S(\Delta A), A \parallel B$. Therefore, the following rule (corresponding to $\Delta A \prec A$) leads from winning-positions to winning-positions:

$$(O_\Delta) \qquad S(\Delta A), A \parallel B \Rightarrow S(\Delta A) \parallel B\,.$$

In order to defend a thesis $\Delta B$, if a system $S(\Delta A_1, \ldots, \Delta A_n)$ of hypotheses is given, the proponent may — if $\Delta B$ is attacked — try to defend $B$ with the system $A_1, \ldots, A_n$ as hypotheses.

This gives a new version of the Rule of ARISTOTLE:

$$(P_\Delta) \qquad A_1, \ldots, A_n \parallel B \Rightarrow S(\Delta A_1, \ldots, \Delta A_n) \parallel \Delta B\,.$$

This GENTZEN-calculus for modal logic is consistent and complete with respect to the given interpretation of $\Delta$. It is known from the literature (e. g. K. SCHÜTTE, Vollständige Systeme modaler und intuitionistischer Logik, 1967) that this GENTZEN-calculus is equivalent to VON WRIGHT's calculus $M'$, if the restriction not to iterate $\Delta$ is omitted.

That our interpretation leads to a calculus already well-known in the modern formalistic literature on modal logic will — I hope — give no additional authority to the interpretation. It has to be

jugded by its own reasonableness. In order to improve this reasonableness, I would like to sketch, how the interpretation can be applied to the traditional problems of modal syllogistics.

First of all, we have to decide how we are to interpret necessary universal affirmative propositions. In ordinary English we can say "All $S$ are necessarily $P$" (e. g. "All men are necessarily mortal").

What is the form of such sentences in modal logic? Superficially there seem to be two different forms appropriate:

$$(1)\ \Delta \bigwedge_x . x \varepsilon S \rightarrow x \varepsilon P . \quad [\Delta S a P] ,$$
$$(2)\ \bigwedge_x . x \varepsilon S \rightarrow \Delta x \varepsilon P . \quad [S a \Delta P] .$$

ARISTOTLE uses in his modal syllogistics very often the first ferm, but sometimes (e. g. Anal. I, 1, ch. 9) he uses the second. We know that this was the main objection of THEOPHRAST: he insisted on using only the first form.

With the help of our interpretation, it is easy to see that only THEOPHRAST's point of view leads to an adequate solution of the problems of modal syllogistics. Already the necessary universal negative proposition is convertible only in the first form (cf. Anal. I, 1, ch. 3)

$$\Delta \bigwedge_x . x \varepsilon S \rightarrow x \varepsilon' P . \ [\Delta S e P] .$$

The second form, whether read as $\bigwedge_x . x \varepsilon S \rightarrow \Delta x \varepsilon' P.$ or as $\bigwedge_x . x \varepsilon S \rightarrow \neg \Delta x \varepsilon P.$ is not convertible.

ARISTOTLE's reasoning with the second form is mostly sound, e.g. if he infers $S a \Delta P$ from $S a M$ and $M a \Delta P$, but on the interpretation here proposed, the second form is obviously not intended by ARISTOTLE: If ARISTOTLE asserts that all men are necessarily living beings (Anal. I, 1, ch. 9) he does not refer to a system $\Sigma$ such that for all proper names "$x$" of men: $\Sigma$ implies $x \varepsilon$ living.

But he refers to a system $\Sigma$ such that $\Sigma$ implies "all men are living beings".

In arithmetic we may have systems $\Sigma$ (e. g. the system of Peano-Axioms) such that $\Sigma$ implies the formulae $A(n)$ for all numerals $n$ without implying the formula $\bigwedge_x A(x)$ —that is the famous case of $\omega$-incompleteness proved by GÖDEL — but ARISTOTLE never deals in his syllogistics with systems in which *all* proper names of his objects occur. He bears in mind systems of predicator-rules given as "necessary" (in my terminology they

would have to be called "analytic"). With respect to such systems $\Sigma$ only the first form $\Delta\, S\, a\, P$ makes sense.

The second main difference between Aristotelian and Theophrastean modal syllogistics is a verbal one. It is concerned with modalities definable in terms of $\Delta$. With the negation $\neg$ only, we may define $\Delta'$, $\nabla$, $\nabla'$ as follows:

$$\Delta' A \leftrightharpoons \Delta \neg A$$
$$\nabla' A \leftrightharpoons \neg \Delta A$$
$$\nabla A \leftrightharpoons \neg \Delta \neg A$$

These four modalities form a modal square

$$\begin{tikzcd} \Delta \arrow[r, dash] \arrow[d] \arrow[dr, dash] & \Delta' \arrow[d] \arrow[dl, dash] \\ \nabla & \nabla' \end{tikzcd}$$

$\Delta$ and $\Delta'$ are contraries, $\Delta$, $\nabla'$ and $\Delta'$, $\nabla$ are contradictories. Therefore we have the two implications

$$\Delta \prec \nabla \qquad (\text{i. e. } \Delta A \prec \nabla A)$$

and

$$\Delta' \prec \nabla' \qquad (\text{i. e. } \Delta' A \prec \nabla' A)$$

If we construct conjunctions of these four modalities, we got

$$\between A \leftrightharpoons \nabla A \wedge \nabla' A$$

as a new modality.

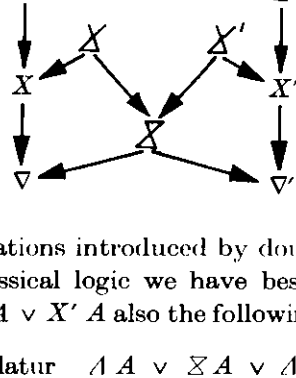
ARISTOTLE uses the Greek term ἐνδέχεται normally in the sense of $\between$, only occasionally in the sense of $\nabla$. THEOPHRAST restricts ἐνδέχεται to the sense of $\nabla$. In the following I'll translate

$\nabla$ as possible
$\between$ as contingent.

If we add (as ARISTOTLE and THEOPHRAST did) the "simple" truth $X$ and falsity $X'$ to the modalities we get by conjunction

$$\between A \leftrightharpoons X A \wedge \nabla' A \qquad \text{(contingently true)}$$
$$\between' A \leftrightharpoons X' A \wedge \nabla A \qquad \text{(contingently false).}$$

This gives nine modalities with implications as indicated in the following figure




We omit the complications introduced by double negation in non-classical logic. In classical logic we have besides the well-known tertium non datur $X\,A \vee X'\,A$ also the following valid adjunctions:

quartum non datur $\quad \Delta\,A \vee \overline{X}\,A \vee \Delta'\,A$
quintum non datur $\quad \Delta\,A \vee \underline{X}\,A \vee \underline{X}'\,A \vee \Delta'\,A$ .

The task of modal syllogistics is a generalization of the task of assertoric syllogistics. There we have to investigate the implications

$$S\,\varrho\,M \wedge M\,\sigma\,P \prec S\,\tau\,P$$

with $\varrho$, $\sigma$ and $\tau$ as variables for the relations $a$, $e$, $i$, $o$. We add the converse relations $\tilde{a}$ and $\tilde{o}$, so that we can restrict ourselves to the above standard form (in contrast to the traditional 4 figures).

We denote the standard syllogisms for short as

$$\varrho \mid \sigma \prec \tau$$

(e. g. $a \mid a \prec a$ for the modus barbara)

In modal syllogistics we have the standard form

$$\Phi\,S\,\varrho\,M \wedge \Psi\,M\,\sigma\,P \prec \Omega\,S\,\tau\,P$$

with $\Phi$, $\Psi$, $\Omega$ as variables for the affirmative modalities $\Delta, X, \nabla, \overline{X}, \underline{X}$.

We abbreviate the standard syllogisms as

$$\Phi\,\varrho \mid \Psi\,\sigma \prec \Omega\,\tau$$

(e. g. $\Delta\,a \mid \Delta\,a \prec \Delta\,a$ for the modus barbara of the class $\Delta\Delta\Delta$).

Altogether we have $30^3 = 27\,000$ standard syllogisms. In spite of this large number it turns out rather easily that there are exactly 567 valid standard syllogisms.

On the basis of Aristotle's Rule each valid assertoric syllogism $\varrho|\sigma \prec \tau$ — and there are 21, as well-known, in standard form — immediately yields the validity of

$$\Delta\,\varrho \mid \Delta\,\sigma \prec \Delta\,\tau\,.$$

Modal syllogisms of this form will be called "of the class $\Delta\Delta\Delta$". By contraposition one gets 21 valid syllogisms for each of the classes $\Delta\ \nabla\ \nabla$ and $\nabla\ \Delta\ \nabla$.

Adding $X$ we have the assertoric class $XXX$ and we get the following "multiplication table" for modalities:

| | $\Delta$ | $X$ | $\nabla$ |
|---|---|---|---|
| $\Delta$ | $\Delta$ | $X$ | $\nabla$ |
| $X$ | $X$ | $X$ | |
| $\nabla$ | $\nabla$ | | |

This table of 6 classes satisfies the "regula peiorem" which was upheld by THEOPHRAST against ARISTOTLE.

In this table we may strengthen the premisses $X$ by $\bar{X}$ (this gives 5 classes) and the premisses $\nabla$ by $\bar{X}$, $\bar{X}'$ or $\bar{X}'$ (6 classes) we may weaken the conclusion $\Delta$ by $X$ or $\nabla$ (2 classes) and the conclusion $X$ by $\nabla$ (8 classes). Altogether we get 27 classes, i. e. $27 \cdot 21 = 567$ valid syllogisms in standard form.

To show that 567 is the exact number requires ruling out all other classes. For example the classes $\nabla\ X\ \nabla$ and $X\ \nabla\ \nabla$ are ruled out as they are reducible—by contraposition—to $\Delta\ X\ \Delta$ and $X\ \Delta\ \Delta$. E. g. no implication

$$X\ S\ \varrho\ M \wedge \Delta\ M\ \sigma\ P \prec \Delta\ S\ \tau\ P$$

is valid, as $S\ \tau\ P$ is not implied by $M\ \sigma\ P$ alone.

There is no philosophical importance attached to the number 567. But with the help of a complete account of modal syllogistics the history of logic (and metaphysics) can be looked at in a new way. In particular, all the traditional talk about "absolute necessity" where some modal logic is used (and at the same time the above definition of relative necessity is rejected) becomes, to say the least, still more suspect. But, of course, it is easy to interpret "absolute" necessity as relative to a system $\Sigma$ of a priori truths.

The inventions of modal calculi in the style of LEWIS on the other hand, can no longer be claimed to be of any help in understanding ARISTOTLE (or his followers). Modalities are of no use at all in mathematics; if they occur in texts of mathematics, they occur only as a sloppy use of language. So it seems as though there is no place for modalities in a critical society in which no system of sentences is accepted without question. But this applies only to "physical" modalities (the so-called "ontic" or even

"ontological" modalities) which occur in the context of descriptive (indicative) sentences.

The situation changes, however, when we consider "ethical" modalities, those called "deontic" or "deontological". (For the rest of the paper, the physical modalities will be denoted with a subscript "$o$" e. g. $\Delta_o$, $X_o$, $\nabla_o$.) Even the most enlightened society depends on generally accepted universal norms which prescribe actions.

I will render unconditional imperatives (such as "Go away!" and "John, get some water!" in ordinary English) in the standard form of an indicative sentence with! prefixed, e. g. "! John gets some water." ("!" may be read as "please.") Let $\Sigma$ now be a system of such imperative sentences! $B_1, \ldots, !B_n$. If the system $B_1, \ldots, B_n$ of indicative sentences logically implies a sentence $B$, I propose to say "$\Delta_\Sigma B$", where $\Delta$ now is an ethical modality (in English, e. g., "obligatory" or "required"). We especially get $\Delta_\Sigma B_1$, $\Delta_\Sigma B_2, \ldots, \Delta_\Sigma B_n$. Modal logic comes in if we omit $\Sigma$ in these sentences and rewrite them as $\Delta B_1, \ldots, \Delta B_n$. The above sentence $\Delta_\Sigma B$, restated as $\Delta B$, now is logically implied (in the sense of modal logic). The system of modal sentences $\Delta B_1, \ldots, \Delta B_n$ therefore may replace the original imperatives. I propose to call them "norms".

Legal authority need not issue imperatives, but may instead, as has been done since the beginning of law, use modal sentences, i. e., norms. This has the advantage that conditional norms such as $A_1 \rightarrow \Delta B_1$, $A_2 \rightarrow \Delta B_2, \ldots$, etc., may be used instead of archaic unconditional ones without causing logical difficulties (if we ignore the difficulties of modern logicians). Modal logic tells us which further conditional norms are logically implied.

The modalities $\nabla$ and $\mathbf{X}$ may be introduced in ethical modal logic as before:

| | |
|---|---|
| $\Delta' A \leftrightharpoons \Delta \neg A$ | (forbidden) |
| $\nabla' A \leftrightharpoons \neg \Delta A$ | (negatively permitted) |
| $\nabla A \leftrightharpoons \neg \Delta' A$ | (permitted) |
| $\mathbf{X} A \leftrightharpoons \nabla A \wedge \nabla' A$ | (discretionary) |

Ordinary moral talk usually is restricted to $\Delta$ ("required"), $\Delta'$ ("forbidden") and $\mathbf{X}$ ("discretionary" in the sense of "neither forbidden nor required"), because no logical implication holds between them.

The distinctive features of ethical modal logic, in contrast to physical modal logic, are the following: 1. there is no ethical modality corresponding to the simple truth $X_o$, 2. the fact that basic norms have to be issued by lawgivers makes it essential, to establish norms for norm-giving, i. e. to introduce iterated modalities.

I'll first treat simple truth $X_o$ in ethical modal logic. If some action should be done, it does not follow that this action will be done.

While in physical modal logic, necessity implies truth:

$$\Delta_o A \prec A$$

we have to omit this implication in ethical modal logic. For defending a modal thesis (some modal hypotheses given) we may still use the attack-defense rule

$$\Delta A \quad | \qquad ? \quad | \quad A$$

but, if the opponent has put a $\Delta$-formula $\Delta A$, he is not obliged to admit $A$. This means, that the dialogue is restricted by the following additional rule

*$\Delta$-attack-rule:* The proponent may not attack any $\Delta$-formula.

The *$\Delta$-defense-rule* will be used for ethical modalities with the same wording as for physical modalities. The Gentzen-calculus for winning-positions, therefore, will only contain the "Rule of ARISTOTLE" ($P_\Delta$), but no longer the rule ($O_\Delta$).

As a formal system ethical modal logic is a proper subsystem of physical modal logic as long as modalities are not iterated.

An example of an implication between modal sentences, valid in physical modal logic, but invalid in ethical modal logic is the following:

$$\bigwedge_x \Delta\, p\, x \prec \nabla \bigwedge_x p\, x\,.$$

The proof in physical modal logic is obvious: from

$$\Delta_o\, p\, x \prec p\, x$$

we get

$$\bigwedge_x \Delta_o\, p\, x \prec \bigwedge_x p\, x$$

which leads with the help of $A \prec \nabla_o A$ to the above theorem.

In ethical modal logic — even if we add as a hypothesis that no contradiction (i. e. no logically false sentence $\Lambda$) is obligatory — $\nabla \bigwedge_x p\, x$ cannot be defended on the basis of $\bigwedge_x \Delta\, p\, x$:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | $\neg \Delta \wedge$ | | | | | | |
| 2 | | $\wedge_x \Delta\ p\,x$ | | | $\neg \Delta \neg \wedge_x p\,x$ | | |
| 3 | | $\Delta \neg \wedge_x p\,x$ | ? | | $\Delta \wedge$ | ? | 1 |
| 4 | | | ? | | | $y_1$ ? | 2 |
| 5 | | $\Delta\ p\,y_1$ | | | | $y_2$ ? | 2 |
| 6 | | $\Delta\ p\,y_2$ | | | $\wedge$ | | |
| 7 | | | ? | | $\wedge_x p\,x$ | ? | 3 |
| 8 | | $y_3$ | ? | | $p\,y_3$ | | |
| 9 | | | ? | | | | |

The proponent has to give up—and, obviously, there is no essentially different strategy available.

(In metamathematics this situation is well-known: there are consistent axiom-systems which are $\omega$-inconsistent, i. e. $p\,x$ is derivable for all $x$ and $\neg \wedge_x p\,x$ is derivable, too).

Though it is trivial that there is no logical implication from $\Delta$ (obligatory) to $X_o$ (true)—this is nothing else than HUME's gap between "ought" and "is" in the other direction—it is more difficult to argue that there is no ethical modality $X$ *corresponding* to $X_o$.

In the context of physical modalities $X_o$ is characterised by the following implication

$$\neg X_o A \prec X_o \neg A$$

(we have $\Delta_o \neg A \prec \neg \Delta_o A$ in this direction only).

The problem, therefore, is to investigate the possibility of an ethical modality $X$ such that $\neg X A \prec X \neg A$. Ordinary language very often uses the words "good" or "right" such that for all actions $A$, if it is not good (or right) to do $A$, then it is good (or right) to omit $A$. Also the phrase "the right thing to do" indicates, that for each action $A$ either to do $A$ or not to do $A$ has to be right.

But the proposed interpretation of ethical modalities starting with a system $\Sigma$ of basic norms, does not yield such an ethical tertium non datur (even if classical logic is applied).

If we start with a system $\Sigma_1$, it may follow

$$\Delta_{\Sigma_1} A \vee \Delta_{\Sigma_1} \neg A$$

for a certain class of actions $A$, but this tertium non datur will never hold for *all* $A$ (because $\Sigma_1$ has to be finite). We may enrich $\Sigma_1$ in order to get a refined system $\Sigma_2$, in which for *more* $A$'s

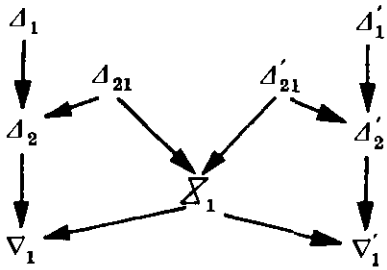
$$\Delta_{\Sigma_2} A \vee \Delta_{\Sigma_2} \neg A$$

is valid—but still $\Sigma_2$ will not be complete: some actions will remain "discretionary", i. e. neither required nor forbidden.

The most important case of ethical modalities with respect to two different bases $\Sigma_1$, and $\Sigma_2$ is the case of a moral code $\Sigma_2$, part of which is made (by institutional enforcement) a legal code $\Sigma_1$.

If we write for short $\Delta_1$ instead of $\Delta_{\Sigma_1}$ and $\Delta_2$ instead of $\Delta_{\Sigma_2}$ (and using $\nabla_1$, $\Xi_1, \ldots$ correspondingly) the assumption $\Sigma_1 \subset \Sigma_2$ yields the following implication $\Delta_1 A \prec \Delta_2 A$. Other implications follow.

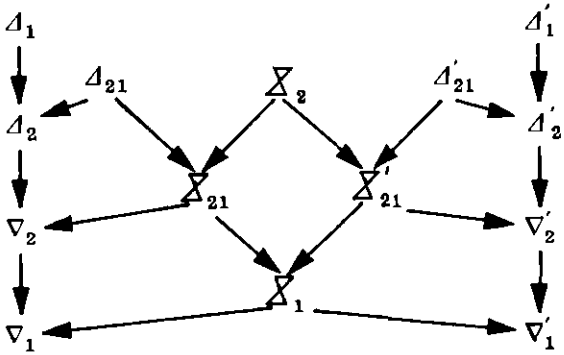
We may easily select 9 modalities with the same implications as in physical modal logic:




Here $\Delta_{21}$ has to be defined by

$$\Delta_{21} A \leftrightharpoons \Delta_2 A \wedge \nabla'_1 A .$$

The similarity with the physical modalities is only a seeming one, because the whole picture (with $\Xi_{21} A \leftrightharpoons \nabla_2 A \wedge \nabla'_1 A$) looks as follows:




Only a sixtum non datur

$$\Delta_1 A \vee \Delta_{21} A \vee \Xi_2 A \vee \Delta'_{21} A \vee \Delta'_1 A$$

is (classically) valid. The mistake to omit $\Sigma_2$ is the mistake of moral rigorism in which all actions are either "good" or "bad".

Modal logic with respect to more than one basis has to be distinguished from modal logic with iterated modalities. The most important use is a "constitution" which restricts the lawgiver (e. g. the English king). Such a constitution forbids certain laws, which are called "unconstitutional" according to the formula: "Congress shall make no law . . . ".

In the administration there are similar situations in which a higher authority restricts the norms of a lower authority. There may be a hierarchy of such authorities. In order to interpret a modal sentence with $n$-times iterated $\Delta$, we have to assume a chain of $n$ authorities (cf. O. BECKER, Untersuchungen über den Modalkalkül, Meisenheim 1952).

The dialogical rules for ethical modal logic with iterated modalities are the same as without iteration.

Under the assumption that all authorities are logically consistent we have $\Delta \prec \nabla$ on each level and therefore, e. g.

$$\Delta\,\Delta \prec \nabla\,\Delta \prec \nabla\,\nabla\,.$$

But e. g. the implication $\Delta\,\Delta \prec \Delta$ is equivalent with the assumption that the lower authority actually requires what it is required to require (by the higher authority).

## 7. *Foundations of Practical Philosophy*

In the lectures up to now, it seems to me, everything has been "plain sailing". No one has any serious difficulties with elementary sentences, with descriptions or abstractors, with the logical particles or modal rules, with rules for predicators or constructive rules for arithmetical symbols, or even with ideal norms in protophysics. Of course, most people still insist on doubting every step, but this is only a kind of philosophic amusement and the practice of the scientists is not disturbed by such verbal games.

In this sense, then, everything has been plain sailing. But now, in this lecture, we are going to enter upon the rough open sea. For I am going to talk about "practical philosophy", and this term, in my usage, stands for moral philosophy in a broad sense, including ethics, legal philosophy and politics.

The very moment I utter the word "morals" I am aware that the direction I have chosen to go will lead me into a barrage of criticism from those who wish to go in another direction. If I were to call attention to the vanity of all our lives, if I were to say that every effort is in vain, that only death is certain and that life itself is the vanity of vanities (you would also have to recognize my existentialist tone of voice), then many of you would accuse me of platitudes. You would criticize me because everyone already knows the truth of sentences like „All men are mortal".

If I went on to ask you whether it might be the case that you have only *knowledge* but not *understanding* of your mortality, rather than answering me you would probably ask yourself: "What is he trying to do? Is he going to try to change our lives? What feeling is he attempting to express? Why does he misuse a lecture in this way? If he wants to talk about moral philosophy, then he ought to aim at exactness, objectivity and truth, as everyone who wishes to do decent philosophy!"

But is "Attempt truth—and nothing else!" a justifiable norm for the moral philosopher? To ask this question is already to deny the imperative, for the question asks for the justification of the norm—not its truth. In this lecture I hope to show that there are non-empirical truths in practical philosophy. But my main purpose will be to show that practical philosophy deals not only with *truth*, but also with *the justification of norms*.

In order to do this, I will have to formulate two "super-norms", called "principles", to be used in justifying norms. Obviously, I will not be able to "justify" these principles, since the term "justification" makes sense only after one has accepted such principles. So, if one defines "faith" in a negative sense, as "the acceptance of something which is not justified", the acceptance of these principles may be called an act of faith. But this act of faith will be different from acts of faith in traditional religions, if we take "religious faith" as a yielding of one's critical judgment before a (religious) tradition.

Let me make a more harmless confession: I cannot find any justification for that playing with the term "moral" which is commonplace in contemporary philosophy. Even the briefest history of the term "moral philosophy" shows that at least up to THOMASIUS, a younger contemporary of JOHN LOCKE, moral philosophy always included attempts to justify the law, i. e., to justify legal norms.

Moreover, the non-religious norms which are part of the Decalogue—Do not kill (other humans)! Do not commit adultery! Do not steal! Do not bear false witness!—are still the core of penal codes all over the world today. Theoretically you may consider this to be merely an historical fact, but since, in practice, you are sitting in this room and net in a prison, I have to assume that you are follewing these norms (with the possible exception of adultery, which, interestingly, has not been considered a crime in England since CROMWELL).

I take legal norms to he a species of moral norms—their specific difference being their mode of enforcement. I understand that the element of puhlic force in the law justifies quite a lot of theorizing in jurisprudence, especially since the political revolutions of modern times no longer permit us simply to rely on the will of the king. But this does not justify our forgetting that legal norms, i. e., enforced moral norms, provide us with a rich set of examples of moral norms. So the term „moral norm" is already well determined, though only by examples.

Thus we find ourselves in the situation of having already accepted certain moral norms. The question now is: "Why do I accept such norms?" It is, therefore, with this question that moral philosophy begins; and it begins, therefore, as normative ethics. To say that, however, is not to say that I will use words uncritically. But my immediate concern is not with a meta-ethical examination of the moral talk of other people. Rather, I will be very much concerned with my own ways of talking with you.

It is now time to get down to business.

The first task of practical philosophy is the reconstruction of a minimum of vocabulary so that we can argue for or against the acceptance of norms. As we will see later, this construction has already been introduced "implicitly" into our natural language, under the influence of philosophical doctrines; and this is the reason for calling a new constuction a "reconstruction". In my earlier lectures I have already argued for some norms, but so far I have limited my concern to norms for using words; now we will be concerned with norms for actions.

One can introduce terms by exemplary determination alone, i. e., by using appropriate situations and exercises. But when I argued for the rules of the dialogical game, I had to use some language in which to formulate the rules. In such cases we need to distinguish between the language used for teachers only and

the language to be taught, e. g., the logical particles $\wedge$, $\vee$, $\rightarrow$, $\bigwedge$, $\bigvee$, $\neg$. I would like to introduce the term "ortholanguage" for that language which is to be taught methodically and "paralanguage" for that language which may be used in explaining how to teach the ortholanguage. Pedagogically a paralanguage will precede the ortholanguage and will be used primarily to describe a situation in which ortho-linguistic terms can be appropriately introduced. Since such descriptions must not presuppose an understanding of those terms which need to be taught, the paralanguage used must not contain any synonyms for these terms.

In this final lecture, therefore, we have to construct an ortholanguage which can be used in practical arguments. If practical philosophy is to equip us with a vocabulary with which we can argue for or against norms, it must first provide us with an ortholinguistic "mental" terminology. In my preceding lectures I used such terms as "want", "know", etc., rather freely. For example, I used paralanguage to say that lens grinders *want* their surfaces to be homogeneous and that I did not *know* of any other approach to geometry which is not *arbitrary*. Now we must dismiss all such "mental terms" from the *para*language so that we can reintroduce them methodically into an *ortho*language.

Let me begin with the term "mind", i. e., with giving instructions for teaching the use of the phrase, "$S$ has a mind". I will take this sentence as an expression which summarizes three more specific "states of mind", namely, "$S$ has the volition to do $A$", "$S$ wants that $X$", and "$S$ has the opinion that $X$". Here "$S$" stands for proper names of persons, "$A$" stands for action-predicates, and "$X$" stands for (descriptive) sentences. In giving these English "translations" of the three mental terms which are to be introduced, I am only giving a hint to those who know English. Officially, so to speak, I must introduce three phrases, namely, $S \blacktriangleright A$, $S \blacktriangleright X$, and $S \rhd X$, without giving either English synonyms for $\blacktriangleright$, $\blacktriangleright$, or $\rhd$ or German synonyms as "Wollen", "Begehren", "Meinen".

I will now give instructions for the use of $\blacktriangleright$, $\blacktriangleright$, and $\rhd$ in a paralanguage without using synonyms. The instructions have to be given in concrete situations in which the use of the terms is appropriate. Here, in a lecture, I can only describe to you how a teacher should instruct his student. The teacher himself should use ortholanguage only, but, in order to describe his teaching, I will have to use paralanguage.

Since imperatives are one of the simplest kinds of talk, the teacher may already have introduced his student to such expressions as "Get some water!" Imperatives refer directly to practice (experience and action), either in the present or in the future. In a situation in which no action has yet been settled on the teacher can begin to develop the art of discussing plans for future actions. Such a discussion does, of course, center upon proposed action, but any contemplated action must wait upon the outcome of the discussion which therefore may be called "epipractical talk". Plans can be talked over without the use of mental terms.

In a discussion the proposal, "Let us do $A$", may have been offered and agreed upon. Then the student will do $A$. However, there may be circumstances such that it is not possible to do $A$ before the next day, and then those who are discussing plans may have agreed to the proposal, "Let us do $A$ tomorrow".

The student has to learn that, before doing $A$, he may describe himself (i. e., his "mental state") by saying, "I $\blacktriangleright A$". If the next day comes and he does not do $A$, he has to learn to say, "I $\blacktriangleright' A$" (with $'$ for negation). He has "changed his mind". If the student participates sensibly in the discussion of plans, he will gradually learn how to use the term "$\blacktriangleright$". If, for example, in the course of a discussion he changes his acceptance into a rejection, one could tell him that at some moment before the linguistic act of rejecting a plan, he must have "changed his mind", i. e., must have changed from $\blacktriangleright$ to $\blacktriangleright'$.

The student needs to learn that the phrases "$\blacktriangleright A$" and "$\blacktriangleright' A$" do not refer to the *linguistic* acts of agreeing or disagreeing to a proposed plan $A$. "I $\blacktriangleright A$" does not stand for the sentence, "I have agreed to do $A$", and the instruction for "$\blacktriangleright$" is not reducible to a definition of the kind: "$S \blacktriangleright A \leftrightharpoons$ If asked, $S$ would agree to do $A$". I take such a *counterfactual* as saying that $S$ has the *volition* to manifest externally his assent to do $A$ under the condition that he is asked. The counterfactual, thus interpreted, is of the form: "$S \blacktriangleright$ (if $B$, then $C$)" and is thereby reducible to "$\blacktriangleright$", but "$\blacktriangleright$" is not reducible to the counterfactual. I am introducing "$\blacktriangleright A$" and "$\blacktriangleright' A$" as referring to the student's "mental" activity, which he can perform without saying anything, perhaps without even having been asked anything.

I propose to use "$S$ has the *volition* to do $A$" in philosophical English, instead of "$S \blacktriangleright A$".

The term "intention" may be used only in those special situations in which someone has the volition to do $A$ *in order* to bring about (to cause) a certain state of affairs, $X$. If it is agreed that certain actions will cause the state $X$, then "to intend that $X$" can be defined as "to have the volition to cause that $X$".

There are several other terms which can be usefully introduced, because they facilitate talking about $\blacktriangleright$. I propose to compare the "mental state" of having a volition with the (ordinary) state of having an object, especially an artifact (a statue, for example). This analogy allows us to speak of the act of forming a volition as:

$$\text{deciding} \leftrightharpoons \textit{forming a volition}$$

and of preparing a decision (i. e., the act of deciding) as:

$$\text{considering} \leftrightharpoons \text{preparing a decision.}$$

Those who participate sensibly in a discussion have to consider various alternatives before deciding what to do. The student has to learn such "considerations" by participating in discussions. Different proposals are formulated in sentences like "Let us do $A_1$" and "No, let us do $A_2$". Discussions also often involve statements to the effect "If we do $A$, we will cause that $X$", and "Let us cause that $X_1$", and "No, let us cause that $X_2$".

I will now describe how the teacher can instruct his student in the use of another basic mental term, "$\blacktriangleright$", in connection with such considerations. "$S \blacktriangleright X$" shall be introduced in such a way that it partially reconstructs the English phrase, "$S$ wants that $X$", so that if $S$ has the volition to cause that $X$, we will say, "$S \blacktriangleright X$". But "$S \blacktriangleright X$" *should not be construed as being synonyms* with "$S$ intends that $X$". Rather, if in the course of a discussion $S$ has proposed a plan and after having unsuccessfully tried to defend it, he finally adopted another plan, then $S$ has to learn that his mental state before that decision is described as "wanting that $X$", though definitely not as "intending that $X$".

In the course of a discussion, therefore, we can learn what it is that we want. Our wantings become "manifest"; and other people can infer from our talking what those wantings are (only, of course, if the possibility of lying, pretending, etc., is reasonably excluded).

In order to explain our dreams (or actions which do not make sense on the basis of our manifest wantings), we may talk of "latent" wantings. For the purpose of constructing a terminology

which is minimally sufficient for formulating the "principles" of practical philosophy, however, this particular extension is not relevant. It is enough that we now can say that the considerations which precede a decision take different wantings into account and finally end in a decision in favour of one over the others.

Remarkably, all such terms as "pain" and "pleasure", "to enjoy" and "to suffer"—I'll call these "hedonistic" terms—are superfluous for our purpose! If someone wants to do A and has tried to argue for it by saying, "I enjoy to do A", he might just as well have simply repeated that he wants to do A. If, instead, he says that everyone in a similar situation finds to do A enjoyable, then, in effect, he is asserting that everyone in such a situation *should* want to do A or at least *may* want to do A. But then he is asserting a modal sentence, and further discussion would have to concern this norm and its possible justification.

Instead of asking whether someone wants something or not, it is common practice to ask him how he "feels" about it or whether or not he "feels happy" about it. But in the reconstruction I propose, these are only different ways of saying the same thing. If someone asserts that he would be "happy" with $X$, the questions still can be asked: *Should* he be "happy" with it? *Should* he not want something else?

One may roughly classify as "hedonistic" all those moral theories which take the wantings of people as merely *given*. Such theories elaborate on ways of harmonizing, reconciling, optimizing, etc., the given wantings (or "interests" as they often are called) in common volitions. *But moral philosophy has the task of formulating principles which allow us to work on merely subjectively "given" wantings and so to discipline them that our decisions about how to act are justifiable.*

Quite a lot of this work is done by working on the "facts" involved in a decision. Practical discussions include not only proposals for action but also theoretical—especially factual—questions. We have to look for a description of our situation, including the "facts" about ourselves, in contrast to merely imaginary "opinions" that we may happen to hold about ourselves. In order to reconstruct the mental term "opinion", the teacher of the ortholanguage has to instruct his student about the use of "$S \rhd X$", i. e., "$S$ has the opinion that $X$". If "$X$" refers to a future state, I propose to use the term "belief" in philosophical English, instead of "opinion".

The term "$\triangleright$" can be introduced as a special case of $\blacktriangleright$. We have already seen that we can distinguish between the linguistic act of assenting to a proposal for action and the mental decision to do so. Here, instead of a volition to do $A$, we have the special case of a volition to assent to a (descriptive) sentence, $X$. So we may introduce the phrase, "$S \triangleright X$", as an abbreviation of "$S \blacktriangleright$ assenting to $X$". But of course the student first has to learn how to participate sensibly in a discussion about the *truth* of $X$. The phrase "$S \triangleright X$" has to be taught and learned within the context of honest discussions. All variants of asserting—such as lying, pretending, etc.—will, unfortunately, be learned quickly enough afterwards.

If we use the same analogy with preparing to form a statue, then forming a statue and finally having it, we now can introduce

judging $\leftrightharpoons$ forming an opinon
thinking $\leftrightharpoons$ preparing a judgment.

The choice of these English terms may seem somewhat arbitrary, but I do not want to lose time discussing their appropriateness. Especially this terminology does not exclude that we often "judge" without any explicit preparation.

In the special case of judging (what to say), we have much better training in doing "the right thing" (namely, in saying the truth) than we have in the general case of deciding (what to do). Very often "the right thing to do" is nothing more than a pseudo-description. But for elementary sentences of the form $S \varepsilon p$, there are already rules for the use of the predicators $p$; at least there are standard examples and counter-examples. In cases such as color blindness—when someone fails to "feel" the same way as other people do about the difference in colour between, e. g., blood and grass,—no common decision procedure is available. Then a quarrel is fruitless, and somo predicators, e. g. "red" and "green" are simply dropped from the discussion.

In our ordinary language we have such "mental" terms as "perception" (i. e., "seeing", "hearing", etc.) and "imagination" (i. e., "memory", "phantasy", etc.). These are used when the truth of a sentence is in question, e.g., "Is this red?" "Yes, I see that it is red." In the case of a past event, memory has to be substituted for perception. I do not deny that the extension of the ortholanguage in the direction of such terms can be useful, but for our purposes here such an extension is irrelevant. We are

pursuing the "philosophy of mind" only to get a minimum vocabulary with which to formulate principles of practical philosophy. It is not necessary for us to reconstruct further "mental terms" now, because we can learn to argue about the truth of practical norms without using mental terms at all.

What *is* of the utmost relevance for practical philosophy is theoretical reasoning. Theoretical reasoning is concerned with the truth of sentences rather than with the ethical modalities of actions. Practical philosophy has the task of finding principles which allow us to argue for or against an action or, as we now can say, for or against a decision. During such practical reasoning we have to take different wantings into consideration, and we have to modify the "given" ones—those which simply happen to occur to us—until we end up with a "justifiable" decision. This means that we have to reject all those wantings which are not "justifiable"; we have to discipline our wantings.

Here the relevance of theoretical reasoning becomes clear: it provides us with a paradigmatic case of how we can and should discipline our wantings. In theoretical matters we are accustomed to restricting our opinons in such a way that we do not insist on opinions which simply happen to occur to us. On the contrary, we try to overcome the imperfections of the opinions with which we begin. Our wantings are more easily disciplined in theoretical matters than in practical matters, precisely because theoretical reasoning is one step removed from action.

Such a program is possible in theoretical matters, because we can critically reconstruct the scientific-humanistic disciplines *(Wissenschaften)*, including a critical reconstruction of their terminology. (To say this is to anticipate a special application of the "cultural principle" which will be formulated later. There we will see that practical reasoning also involves the more difficult job of reconstructing a genesis of concrete situations.) In the scientific-humanistic disciplines we appropriate to ourselves the reasonable use of terminology, and by this method we learn to apply the language in new situations. All mere fancy is methodically excluded; we learn to surrender ourselves to reason.

In philosophical language, the term "subjectivity" is often used to refer to a general unwillingness to surrender one's own opinions. I wish to adopt this term. But an opinion is not to be construed as "merely subjective" only if it is idiosyncratic with the person who holds it. An opinion still may be "merely subjec-

tive" even if it is held by many people. In the sciences it is taken for granted (and in the humanities it should be taken for granted) that the subjectivity of opinion is independent of the number of people who hold an opinion. A sentence does not become true by the mere fact that it is asserted by many people. The many, *οἱ πολλοί*, usually are restricted to subjectivity, whether by their idiosyncracies or by uncritically accepted fashions or traditions, including the traditions of natural language.

It is usual to contrast the mere subjectivity of opinion with the "objectivity" of truth, but, as we have seen in all our investigations into theoretical philosophy, the truth of sentences always has to be a human accomplishment—an achievement of persons. No person can do more than try to overcome his subjectivity, and this is the aim of logical discipline.

It may be helpful to have a term which fits in with our philosophical traditions: so I propose (I am, for the moment, still restricting our concern to the truth of sentences) to call the required overcoming of subjectivity "transcendence of subjectivity"—or "transsubjectivity", for short. This is still subjectivity, but a subjectivity which is aware of its own limits—and tries to overcome them. Transsubjectivity is not a fact, but it is not a postulate either. Transsubjectivity is simply a term characterizing that activity in which we are always already involved if we begin to reason at all.

If we formulate the imperative sentence, "Let us transcend our subjectivity!", we have here a "norm" which makes it possible for us to justify all other norms (norms in the ordinary sense). Transsubjectivity is, therefore, a "supernorm"; it is a "first norm" or, as the Latin term goes, a "principle". We may compare the principle of transsubjectivity to a constitution which makes laws possible. But since this principle is the basis not only of all law but of all morality, I propose to call it, for short, "*the* moral principle" (Das Moralprinzip).

Looked at historically, the moral principle of transsubjectivity is, of course, nothing other than a reformulation of KANT's "categorical imperative". Further, the same crucial question which arises with KANT's "formal" ethics also arises here: even if a person recognizes that transsubjectivity is a necessary condition of moral reasoning, he can still ask whether this one moral principle is also sufficient for moral reasoning. We can answer this question only

by examining how we can come to a justifiable practical decision in a concrete situation.

In order to reach a decision in a particular situation, we have to apply "material" norms, and we have to justify the material norms which we apply. According to the principle of transsubjectivity, each wanting which is taken into consideration has to be formulated as a (universal) norm: "If the situation is such and such, then the following action is obligatory (or permissible etc.)...". But there will be neither permission nor obligation, unless some wantings are distinguished from others in such a way that their satisfaction is required to be taken into consideration (also by every other person concerned). It may turn out that some elements of the situation finally rule out the satisfaction—but "prima facie" (or as a "rule of thumb") it has to be considered. The principle of transsubjectivity demands that all merely subjective wantings are to be ruled out of consideration. But we still have the problem of distinguishing between those wantings which are and those wantings which are not "merely subjective".

In order to have simpler terms than "merely subjective" and "transsubjective", I propose to revive a rather oldfashioned term. Let me call the merely subjective wantings "inessential". The wantings which are left, even when one tries to be as transsubjective as possible, are then to be called "essential" wantings. To accept a wanting as essential is, thereby, defined as accepting the obligation to take it into consideration. Further, since it may be the case that someone does not actually want what would be an essential wanting if only he did want it, there is reason for introducing another traditionally well-known term, "need" („Bedürfnis"). If, under certain conditions, a wanting to do $A$ would be an essential wanting if it were wanted at all, I propose to say that there is a "need" to do $A$.

Of course, this added terminology does not solve the problem of how we are to decide whether there is a need for a "given" wanting. But we now have a terminology which is adequate for formulating what can be said "in the abstract" (i. e., without dealing with any particular "concrete" situation) about the distinction between essential and inessential wantings. (This claim will have to be justified in the following pages.)

Again we begin with practice. This means, as we have seen, that we are always already involved in the discussion of plans. We are not only "all along" (HEIDEGGER: „immer schon") talking with

one another, but also we are "all along" acting together, cooperating with one another. Some wantings are "all along" accepted as needs. For example, as children, all of us could grow up only because adults took care of our needs. All of us, therefore, know that we are very often in need.

Moreover, we human beings, unlike animals, anticipate our future needs, because we know them *via* language. Therefore, if we try to satisfy our present wantings just as they occur to us, we find ourselves worrying that we might be jeopardizing the satisfaction of future needs.

The needs which we share with animals may be called provisionally "natural" needs; they are essential wantings for us insofar as we are natural beings. But there are also other needs. Now the task of practical philosophy is to establish principles which give us a method of deciding practical questions in our present situation. And the very existence of language—and therefore the existence of norms for our linguistic activities—shows that we cannot consider ourselves as merely natural beings. I therefore propose—once more provisionally—to use the term "cultural" to refer to all essential wantings which cannot be called "natural". So we may speak both of "cultural needs" and of "natural needs".

But it still has to be shown how, in a concrete situation, any wantings—natural or cultural—are to be distinguished as either essential or inessential. One cannot say, *in abstracto*, which norms will be justified in a concrete situation; too much depends upon the particularities of the situation. But we can say, *in abstracto*, that if any norms are uncritically accepted in a concrete situation, there is always the risk of deciding against our needs. The only way open to us by which we can overcome the limitations of our own subjectivity, i. e., of the uncritical acceptance of norms, is the attempt to become more transsubjective.

But how do we know that all this is not mere verbiage? Is it not merely "formal" to repeat the moral principle of transsubjectivity over and over again?

Following the history of practical philosophy which leads from KANT's "categorical imperative" to MARX's "dialectics", I would like to suggest a 'further principle' which provides, still *in abstracto*, the "materia" for the "forma" of the moral principle. This second principle, first of all, requires us to look at the *genesis* of a concrete situation before deciding what to do or not to do.

The term "genesis" can mean the "history" of a situation, i. e., a detailed answer to the question asking how the situation factually came into being. We can call the answer to this question, the "factual genesis" of a situation. Part of the factual genesis is a "causal explanation" of the situation, i. e., an account which offers hypothetical universal laws by virtue of which the given situation can be derived from some—hypothetically assumed—earlier situation.

Historical and causal questions are fully appropriate and sufficient if we are dealing only with natural phenomena. But a concrete situation involving human beings includes not only natural wantings but also cultural wantings. And when culture is relevant, then another kind of genetic question becomes relevant: *Why* was this brought about? What aims were sought? What was achieved and where did the attempts fail? What is asked for here will be called a "normative genesis".

In order to justify "normatively genetic" questioning (or, to be more precise, the imperative to ask such questions") one need only realize that "culture" finally has to be distinguished from "nature" just by being "brought about": it is a human achievement. We can become clearer about what kind of genesis this is if we look more critically at the ways in which cultural needs are discussed. Then we will be able to arrive at a more detailed prescription of an appropriate method for dealing with cultural needs.

Before we can discuss actions in a concrete situation, we have first to be able to "describe" the situation. I will call any such description an "abstract situation". The "abstract situation" serves as a *model* for the concrete situation. But the choice of a model already presupposes an anticipated evaluation of the concrete situation, since the construction of a model involves decisions about whether certain elements in a possible description are "relevant" or "irrelevant". Therefore, I propose that the term "model" be understood as a description of which one claims that it contains only "relevant" elements of the concrete situation.

Here once again it becomes necessary to apply the moral principle of transsubjectivity, for one must be guided by this principle if he is to overcome his own subjective illusions about a situation. Such a guide is especially important if one is himself a relevant part of a situation.

The only way we can avoid arbitrariness in our choice of a model is to begin with the explicit admission that no element of a con-

crete situation shall be ruled out at the beginning. This means that we have to look at each situation as an *open complex*. It is assumed to be a *complex*, because more than one element may be relevant. It is an *open* complex, because the number of relevant elements is initially unrestricted. Of course, in any particular dialogue one can admit only a finite number of elements which are claimed to be relevant, and this means, in particular that only a finite number of wantings can be considered essential.

But to say that a concrete situation is to be looked at as an open complex only serves to remind us that we already have to be transsubjective in our description of a concrete situation. For our purposes now, it is even more important that we realize that we can work with a model only if we look for a "normative genesis" of the concrete situation as it is described by the model of it which we construct. There will be some wantings which will have to be accepted as needs without our going into the question of a normative genesis, and these will be what I have termed "natural" needs. (If some needs were not accepted as natural, there would be no ending of an argument.) So, there will be mainly the problem of deciding which wantings in the complex under consideration are to be accepted as cultural needs.

Schematically, we will have a complex $(C_1, \ldots C_n)$ of wantings which are claimed to be cultural needs. If we assume that the complex $(C_1, \ldots C_{n-1})$ is already justified, an attempt to justify $C_n$ could lead to unending circles; for one might have to rely on $C_n$ as already accepted in order to justify any one of $C_1, \ldots C_{n-1}$. But a "normative genesis" provides a non-circular method of justification (and the only one so far proposed and practiced). The complex $(C_1, \ldots C_n)$ is considered—once more rather schematically—as an abstract situation which has *developed* from a *simpler abstract situation* $(C'_1, \ldots C'_{n-1})$. Here $C'_1, \ldots C'_{n-1}$ must not be identical with $C_1 \ldots C_{n-1}$ but may be an "earlier stage" of the latter. Then a new wanting, $C'_n$, is added to the complex $(C'_1, \ldots, C'_{n-1})$ which is thereby transformed into a new situation $(C_1, \ldots, C_n)$.

Schematically, we have

$$(C'_1, \ldots, C'_{n-1}) \Rightarrow (C'_1, \ldots, C'_{n-1}), \quad C'_n \Rightarrow (C_1, \ldots, C_n)\,.$$

The simplest case of this scheme is

$$(C'_1) \Rightarrow (C'_1), C'_2 \Rightarrow (C_1, C_2)$$

which, since FICHTE, has been known as a "dialectical" triad, leading from a "thesis" $C'_1$ via an "antithesis" $C'_2$ to a "synthesis" $C = (C_1, C_2)$.

The term "dialectical" has been used specifically for that kind of genetic account which is appropriate for the consideration of practical problems. We therefore could use the term "dialectical" for the imperative to look at every concrete situation in just this normatively genetic fashion. The specific character of this super-norm is that we have to *justify* the genesis from $(C'_1, \ldots, C'_{n-1})$ via $(C'_1, \ldots, C'_{n-1})$, $C'_n$ to $(C_1, \ldots, C_n)$. This justification is neither simply an historical report nor a causal explanation; it is a justification guided by the moral principle of transsubjectivity.

As the term "dialectical" nowadays occurs very often as a merely fashionable word without a reasonably fixed usage, I would like to mention, that the term is irrelevant here. It would be sufficient, though more cumbersome, to speak of normative genesis only. As normative genesis distinguishes *cultural from natural* needs, it seems to be appropriate to use the phrase "the cultural principle" (Das Kulturprinzip) for the super-norm to give a normative genesis for norms.

More important, it seems to me, is not to become too rigid in prescribing the schemata of this principle. They may be applied in many variations.

Instead of working out further schemata, I'll give two examples, where the cultural principle is applied. The first example is adapted from CONDORCET.

Since the prehistory of mankind we find nomads forced to be always on their way, because they have to find food for their animals ($C'_1$). On the other hand the tradition of gathering fruits may have developed—say in a group of people who mostly lived by fishing—into a primitive stage of agriculture ($C'_2$). If now at some place a fishing group was joined by a group of nomads, there was a chance of a synthesis: cattle-breeding together with a developed agriculture (such that enough food for the animals was produced) resulted in a new type of *economy*: agriculture ($C$) as we know it since the neolithicum.

Such an example does not belong to philosophy—so the adequacy of this normative genesis is irrelevant for our purposes. But the next example will belong to philosophy itself: the normative genesis of dialogues. Let us start at a stage where there are only

material dialogical games without using logical particles, i. e. dialogical games using only elementary sentences and rules for the occurring predicators ($C_1'$). As a next step we may imagine a primitive use of logical particles ($C_2'$), separated from the dialogical games. The synthesis ($C$) is a stage in which material dialogues with logical particles ($C_1$) and formal dialogues ($C_2$) are fully developed.

The cultural principle *requires* that, before we can come to a practical decision in a concrete situation, we must construct a normative genesis of a model of the concrete situation. And here, a normative genesis is a genesis such that each step of it can be justified transsubjectively.

The only principle guiding our practical judgment still will be transsubjectivity, but now this criterion is set to work on all the material provided by a concrete situation. In order to apply the principle, however, we have to work with abstract situations, i. e., with models, and we have to construct a normatively genetic sequence of such models, *stretching* from a merely "natural" state up to the cultural stages of the concrete situation being considered.

As HEGEL has stressed, it would be a rationalistic illusion for us to believe that the construction of a normative genesis should be carried out without a thorough investigation of the actual history of the concrete situation. Very often only historical knowledge provides us with an adequate "understanding", i. e. a transsubjective justification for cultural wantings which may seem at first to be unjustifiable. Historical knowledge is unique in enabling us to become more transsubjective.

The construction of a normative genesis thereby turns out, normally, to be a partial reconstruction of the history of the concrete situation. Not all historical details are relevant for a normative genesis, but it is a reasonable guess that the essential steps which led to our present situation actually were taken (in a more or less obscure way) during its history.

The construction of a normative genesis has to be done stepwise: we have to start with some norms taken from immediate practice and have to apply these norms for a critical understanding of some parts of history. From history we have to go back to our genetic account of norms, which now—hopefully—can be "improved". Then we have to go to a critical understanding of history again etc.

This movement should not be a circle, but—as in the case of the

famous "hermeneutic circle", too—should be a spiral. As no other term seems to be at hand I would like to call this spiral movement for constructing normatively genetic accounts the "dialectical" spiral. "The dialectical spiral" is not a descriptive term for a way people think, but it is required to think this way in order to satisfy the cultural principle.

The principle of constructing a normative genesis allows us to justify "cultural" needs. At the same time, claims that some wantings are "natural" needs can be refuted—namely by constructing a normative genesis which "explains" those wantings (which are claimed to be "natural") as merely resulting from factual failures to understand the normative genesis. In this way it is the principle of "normative genesis" which allows us to distinguish between cultural needs and wantings, which are merely claimed to be (either natural or cultural) needs.

Together, the moral and the cultural principle prescribe in abstract terms how one can and should prepare practical decisions *in* a concrete situation. They are *not* arbitrary principles; rather, they are the only principles which make "reasonable" decisions possible. The term "reason" can be introduced into the ortholanguage only as its keystone, i. e., at the very end, after the moral and the cultural principle have already been formulated. Insofar as it concerns practical problems, "reason" is called "practical reason" and its meaning can be understood only after the moral and the cultural principle first have been practiced and then have been formulated.

The ortholanguage I have proposed, together with the two principles of practical reason, are all a critical reconstruction of traditions of practical philosophy, which I have carried out as transsubjectively as I could. I have proposed a normative genesis of the abstract doctrine called "practical philosophy", so that the method I have proposed—how to teach the terminology and the principles of practical philosophy—is nothing other than an application of just these principles: the method is identical with its own result.

Normative logic and ethics is, thereby, a "closed system" of *pure philosophy*—but open for all applications.